

Introductory Lectures on D-Branes

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Abstract

This is a pedagogical introduction to D -branes, addressed to graduate students in field theory and particle physics and to other beginners in string theory. I am not going to review the most recent results since there are already many good papers on web devoted to that. Instead, I will present some old techniques in some detail in order to show how some basic properties of strings and branes as the massless spectrum of string, the effective action of D -branes and their tension can be computed using QFT techniques. Also, I will present shortly the boundary state description of D -branes. The details are exposed for bosonic branes since I do not assume any previous knowledge of supersymmetry which is not a requirement for this school. However, for completeness and to provide basic notions for other lectures, I will discuss the some properties of supersymmetric branes. The present lectures were delivered at Jorge André Swieca School on Particle and Fields, 2001, Campos do Jordão, Brazil.

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1 Introduction

The fundamental problem of high energy theoretical physics is to provide us with a description of the intimate structure and interaction of the Nature. At low energies, accessible to particle physics experiments, the accepted models are based on QFT which predicts particle-like excitations of the fields interacting through three fundamental forces: electromagnetic, weak and strong and through a classical gravitational interaction. Nowadays, it is known that the first three interactions can be unified at intermediate energies and it is believed that at a scale of energy around 10^{-33} cm, (the *Planck length*) all the interactions should be consistently described by an unique theory which is unknown yet. There are several arguments in support of the idea that at this scale the excitations (or at least part of them) of the fundamental object of the theory should be string-like rather than point-like and studying such of hypothesis is the object of the string theory. N. Berkovits will review in his lectures at this school the problems of field theory and gravity at the Planck scale and the arguments in favor of string theories. For our purpose, it is enough to say that many people consider string (based) theories as the most promising candidate for the final theory of the Nature basically because they predict all particles and interactions among them, including the gravity, from a basic object which is the string. However, due to the largeness of the unification scale, the predictions and the constructions are theoretical; all what is required is that the theory satisfy some internal consistency conditions and reproduce at the low energy limit the known theories, i.e. QFT and gravity. Results that could be decisively connected with phenomenology have not been rigorously obtained yet.

Despite many of its conceptual successes, string theories suffer from several serious problems which serve as arguments in favor of criticism against them. One of the drawbacks of strings is that they appear in a too large number of theories, all with equal rights to a theory of everything: Type I, Type IIA, Type IIB, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$. All of these theories are supersymmetric and their names are related to the number and type of supersymmetry and of the gauge group. Another problem is that all these theories live in a ten dimensional space-time with one time-like direction. Since the world in which we live has only three space-like directions, one has to explain how the space-time of string theory reduces to the physical one. One way of thinking to the dimensional reduction is by considering that some of the $d = 10$ space-time directions are *compactified*. There are some recent progress in this direction, but no completely satisfactory answer is known at present. However, with the discovery of the D-branes six years ago, many revolutionary ideas about string theories and new angles of attack to the old problems have been emerging.

The *D*-branes are extended physical objects discovered in string theories. One type of *D*-branes, called BPS-branes, which saturate a relation between supersymmetry and

energy, play an important role in establishing relations among the string theories called *dualities*. By a duality, two string theories are mapped into one another. Since the theories can describe the interactions in different regimes (for example on different compact space-time or in weak-strong coupling limits), they can represent relations among these different limits (of string theories.) This points out towards an unique underlying theory of which limits are described by string theories. This unknown yet theory is called M (or, sometimes U)-theory and one of its limits is the supergravity in $d = 11$. Beside their importance in unification of string theories, the D -branes are necessary for their internal consistency. Indeed, in Type II theories there are some massless p-form bosonic fields called Ramond-Ramond (RR) fields after the name of the perturbative sector of string spectra in which they appear. The D -brane are the sources of these fields and carry their charges. In simple situations in which the branes can be considered as hyperplanes, there are similar relations between RR-charges and the topological and the Noether charges encountered in electromagnetism. However, being extended objects, the D -branes can have more sophisticated topologies, in which case there are topological contributions to the charges. Therefore, the classification of charges is given by an extended cohomology called K -theory rather than by the De Rham cohomology. There are subtle relations in K-theory among different D -branes and between branes and $d = 11$ supergravity. BPS as well as non-BPS branes are involved in that and the tachyon fields represent the mechanism that controls the evolution of the system.

Beside their crucial role in understanding string theories, the D -branes have been successfully used to explain various supersymmetric and non-supersymmetric field theories, the entropy of some models of black-holes and, more recently, there have been attempts in understanding some cosmological issues and the hierarchy problem. New developments appear daily on the net, and one of the hot topics is the geometrical structure of space-time and field theories at the Planck scale, which could be non-commutative.

The D -branes represent a quickly developing field, where many novelties appear almost each year. There are many fascinating problems in this direction which has been inspiring in both physics and mathematics. However, the purpose of these lecture notes is by far more modest. I am not going to review nor the successes neither the various theories that are based on and that involve the D -branes. To this end, there are many very good references at <http://xxx.lanl.gov>.

The aim of these lecture is to show to particle physics theorists how some basic properties of D-branes have been computed and how field theory techniques can be used to obtain information about branes. By that, I hope that graduate students and researchers will feel more confident and encouraged to read string literature. Also, I would like to provide a background for the more advanced topics in brane theory that will be presented at this

school by I. Ya. Arafewa, S. Minwalla and K. Stelle.

Due to the pedagogical line that I had chosen and since most of the audience was not familiar with string theory, supersymmetry and supergravity, I focused in my talks on *physical properties* of bosonic D -branes as a low energy effective action, tension and boundary field description. However, some properties of supersymmetric branes also emerged mostly during the discussion sessions, therefore I added a part in which the same basic properties but for the supersymmetric branes are listed.

The structure of these notes is as follows. In Section 2 the basic results of free bosonic string theory are revisited. I will present the massless spectrum of either open and closed bosonic string theory to argue the presence of graviton and other bosonic excitations which will be used in the next sections. In Section 3 we deduce the Born-Infeld action or the low energy action of a D -brane. I work out in detail the case of the $D25$ -brane and the sigma model in a general closed string background. This section is based on In Section 4 we find the tension of the D -branes. In Section 5 we present the microscopic description of D -branes known as the boundary state formalism. In the last Section the same results for supersymmetric branes are presented.

There are many references on strings and D-branes and it would be impossible to mention all of them. I have tried to refer mainly to more advanced lecture notes rather than original papers for pedagogical reasons. The references that I have been used throughout this work have been selected according to my preference.

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2 Review of Basic Results from Free Bosonic String Theory

In this section I am going to review some basic results from free bosonic string theory that will be useful in studying the D-branes. This is a text-book material [1, 2, 4] but it is included here since these ideas might be unfamiliar to some part of the audience.

2.1 Classical free bosonic string theory

The starting point in discussing a classical free string is its action which is proportional to the area described by the string during its evolution in spacetime [1, 2, 3]. The classical action can be cast in a polynomial form known as Polyakov action which, in the conformal gauge and in the Minkowski background, is given by the following formula

$$S = -\frac{T_s}{2} \int_{\Sigma} d^2\sigma \partial_{\alpha} X^{\mu} \partial^{\alpha} X_{\mu}. \quad (1)$$

Here, T_s is the string tension which is related to the Regge slope by the formula $T_s = (2\pi\alpha')$ and X^{μ} are the string coordinates in $D = 26$ dimensional space-time, $\mu = 0, 1, \dots, 25$. The world-sheet is parametrized by a time-like parameter $\tau = \sigma^0$ and a space-like one $\sigma = \sigma^1 \in [0, \pi]$. The two metrics, on the world-sheet Σ and on the space-time, respectively, are taken to be Minkowskian. The action given by Eq.(1) has the following symmetries: $SO(1, 25)$ in space-time, $SO(1, 1)$ on the world-sheet and a residual two-dimensional conformal symmetry [1, 2, 3].

The variational principle applied to the action (1) gives the following equations of motion

$$\partial^{\alpha} \partial_{\alpha} X^{\mu}(\tau\sigma) = 0 \quad (2)$$

which hold only if the boundary conditions are satisfied, too. The closed string boundary conditions simply express the fact that the string coordinates are univalued

$$X^{\mu}(\tau, \sigma + \pi) = X^{\mu}(\tau, \sigma). \quad (3)$$

In the case of the open string, one can impose Poincaré invariant or Poincaré breaking boundary conditions. They are given by the following relations

$$\partial_n X^{\mu}|_{\partial\Sigma} = 0 \quad (4)$$

$$\delta X^{\mu}|_{\partial\Sigma} = 0, \quad (5)$$

which are known as Neumann b.c. and Dirichlet b.c., respectively. The topology of the world-sheet depends on the topology of the string. At classical level, the world-sheet of

the closed string is equivalent to a cylinder or a two-dimensional annulus, while the world-sheet of an *open* string can be continuously transformed into a disk with the boundary corresponding to the circle.

In order to determine the solutions of the equations of motion one can study the variation of the action with fixed string configurations at the initial and final time: $\delta X^\mu(\tau_1) = \delta X^\mu(\tau_2) = 0$. (The boundary conditions may affect the finiteness of the physical quantities derived out of the action but not the solutions of the equations of motion.) For these configurations the boundary conditions are given by the following relations

$$\partial_\sigma X^\mu|_{\sigma=0,\pi} = 0 \quad \text{N.b.c.} \quad (6)$$

$$X^\mu|_{\sigma=0,\pi} = \text{ct.} \quad \text{D.b.c.} \quad (7)$$

There is a constraint in the theory, namely that the energy-momentum tensor vanishes. This constraint can be understood by considering the string coupled to a two-dimensional graviton, i.e. on a curved world-sheet, where it arises as the equation of motion of the two-dimensional gravitational potential. However, in the present case it is a condition that should be imposed on the system by hand [1, 2, 3] and it is give by the relation

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} \eta_{\alpha\beta} \partial_\gamma X^\mu \partial^\gamma X_\mu = 0. \quad (8)$$

The Weyl invariance of the action $h_{\alpha\beta} \rightarrow \exp \Lambda h_{\alpha\beta}$ implies that the trace of the energy-momentum tensor vanishes

$$\text{Tr } T_{\alpha\beta} = T^\alpha_\alpha = 0. \quad (9)$$

The relations given by Eq.(8) and Eq.(9) represent strong constraints on the system and in the quantization process they should be implemented at the quantum level. They reflect the fact that string theory is a conformal field theory in two dimensions, a property that determines all the features of bosonic string physics.

The solutions of the equations of motion can be found by employing the method of separation of variables. The Fourier expansion of closed string solution is given by

$$X^\mu(\tau, \sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} (\alpha_n^\mu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)}). \quad (10)$$

The above relation shows that the most general closed string solution is a linear superposition of right- and left-moving modes with Fourier coefficients α_n^μ and $\tilde{\alpha}_n^\mu$, respectively. Here x^μ and p^μ represent the coordinates of the center of mass of the closed string and their canonical conjugate momenta, respectively.

The boundary conditions of the open string can be chosen either Dirichlet or Neumann on each direction. Therefore, one can have NN, DD, ND and DN solutions of the equations

of motion given by

$$\text{N-N} \quad : \quad X^\mu(\tau, \sigma) = x^\mu + 2\alpha' p^\mu \tau + 2i\alpha' \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma, \quad (11)$$

$$\text{D-D} \quad : \quad X^\mu(\tau, \sigma) = \frac{x^\mu(\pi - \sigma) + y^\mu \sigma}{\pi} - i\sqrt{2\alpha'} p^\mu \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \sin n\sigma, \quad (12)$$

$$\text{D-N} \quad : \quad X^\mu(\tau, \sigma) = x^\mu - i\sqrt{2\alpha'} \sum_{r \in \mathbb{Z}'} \frac{1}{r} \alpha_r^\mu e^{-ir\tau} \sin r\sigma, \quad (13)$$

$$\text{N-D} \quad : \quad X^\mu(\tau, \sigma) = y^\mu + i\sqrt{2\alpha'} \sum_{r \in \mathbb{Z}'} \frac{1}{r} \alpha_r^\mu e^{-ir\tau} \cos r\sigma, \quad (14)$$

where $\mathbb{Z}' = \mathbb{Z} + 1/2$. The closed string solution (10) is Lorentz invariant in $D = 26$ dimensions. The only open string solution which is invariant under $SO(1, 25)$ is (11). The other solutions break the Lorentz invariance down to $SO(1, p) \times SO(25 - p)$.

2.2 Massless spectrum of bosonic open string

Let us see what is the particle content of the free bosonic string theory. To this end one has to quantize the string, but care should be taken since the theory is subjected to the constraints (8) and (9). Consequently, not all degrees of freedom are physical. By implementing the constraints at the quantum level one can remove the effect of the non-physical degrees of freedom. One way of doing that is by employing the canonical quantization method which implies that the constraints will appear as operatorial equations in the Fock space of the theory. Their solutions represent the physical states (for more details on the quantization of the string theory through various methods see [1].)

Consider the Lorentz invariant solution of the open string theory ((12), (13) and (14) are quantized in exactly the same way.) The coordinates X^μ can be viewed as two-dimensional fields, which have the equal-time commutators given by

$$[X^\mu(\tau, \sigma), \dot{X}^\nu(\tau, \sigma')] = i\eta^{\mu\nu} \delta(\sigma - \sigma') \quad (15)$$

$$[X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')] = [\dot{X}^\mu(\tau, \sigma), \dot{X}^\nu(\tau, \sigma')] = 0. \quad (16)$$

The η^{00} component of the Minkowski metric generates negative norm, unphysical states which must be removed from the spectrum. One can obtain the commutation relations among the Fourier coefficients by plugging (11) into (15) and (16). The result is

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu} \delta_{m+n, 0} \quad (17)$$

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}. \quad (18)$$

Note that for $\mu \neq 0$ and $\nu \neq 0$ the relations above are the usual commutation relations of linear oscillators scaled by factor of m from which we see that α_n operators with $n > 0$ play the role of annihilation operators while for $n < 0$ they act as creation operators. Therefore, it is possible to define a vacuum state with respect to these operators and to construct the Fock space. There is no $n = 0$ mode but one define it as being the momenta of the center of mass [1, 2, 3].

The unphysical states corresponding to the time-like direction of space-time are removed by the energy-momentum constraints. In order to implement them on the Fock space, one has to write firstly the Fourier expansion of the components of the energy-momentum tensor [1]. The Fourier coefficients of $T_{\alpha\beta}$ can be expressed in terms of Fourier coefficients of coordinates as follows

$$\begin{aligned} L_m &= \frac{1}{2} \sum_{n \neq 0} \alpha_{-n} \cdot \alpha_{n+m} \\ L_0 &= \alpha' p^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n, \end{aligned} \tag{19}$$

where “ \cdot ” denotes the scalar product in the Minkowski space-time. The Fourier coefficients of the energy-momentum tensor L_m satisfy an algebra called the *Virasoro algebra* which has the following form

$$[L_m, L_n] = i(m - n)L_{m+n}. \tag{20}$$

The Virasoro algebra is the infinite algebra of the generators of the two-dimensional conformal group. The existence of this symmetry guarantees that the system is integrable. However, upon quantization one can see that an anomalous term appears in the algebra (20). This term cancels if $D = 26$.

The physical states form a subspace of the Fock space which is defined by acting on the vacuum with the creation operators. The vacuum $|0\rangle$ is defined by the following conditions

$$|0\rangle \equiv |0\rangle_{\alpha} |p\rangle \tag{21}$$

$$\alpha_n^{\mu} |0\rangle_{\alpha} = 0, \quad n > 0 \tag{22}$$

$$\hat{p}^{\mu} |p\rangle = p^{\mu} |p\rangle. \tag{23}$$

Alternatively, we will use the notation $|k\rangle$ for $|0\rangle$. The first two relations just define the vacuum of the linear oscillators. The last relation tells that the vacuum continues to make sense when it is translated.

The physical states are defined as being those states of the Fock space that obey the constraints coming from the vanishing energy-momentum tensor. One can show that only half of the Virasoro operators should be imposed on the Fock space since the full set of

constraints is incompatible with the Virasoro algebra [1, 2, 4]. The conditions that define the physical states are

$$L_m|\phi\rangle = 0, \quad m > 0 \quad (24)$$

$$(L_0 - 1)|\phi\rangle = 0. \quad (25)$$

The -1 term in the last equation above comes from the normal ordering of the operator L_0 . Since the Fourier coefficients of the energy-momentum tensor L_m are expressed in terms of creation and annihilation operators as Eq.(19) shows, one has to normal order them through the quantization process. The normal ordering affects only L_0 operator by the -1 term as can be easily checked up.

A special class of states are the *spurious states*. These are states that belong to the equivalence classes of physical states. Two physical states are said to be equivalent if

$$|\phi'\rangle \sim |\phi\rangle \Leftrightarrow \exists |\psi\rangle : |\phi'\rangle = |\phi\rangle + |\psi\rangle \quad (26)$$

and $|\psi\rangle$ is a spurious state, i.e. it satisfies the following equations:

$$(L_0 - 1)|\psi\rangle = 0 \quad (27)$$

$$\langle \psi | \phi \rangle = 0, \quad (28)$$

for any physical state $|\phi\rangle$. An important example of a spurious state is the one obtained by the action of the operator L_{-1} on the vacuum:

$$|\psi\rangle = L_{-1}|0\rangle, \quad (29)$$

which is used to show the gauge invariance of the vector state as we will see later. Any state that is a linear combination of L_{-n} operators is a spurious state

$$|\psi\rangle = \sum_m^{\infty} a_m L_{-m} |\chi_m\rangle. \quad (30)$$

The physical states can be classified according to their mass. The masses are the eigenvalues of the mass operator related to the momentum of the particle by the relativistic mass-shell relation: $M^2 = -p^2$. The mass operator is given by the action of $L_0 - 1$ which describes the mass-shell operator. Indeed, from

$$(\alpha' p^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n - 1)|\phi\rangle = 0 \quad (31)$$

it follows that the mass operator is given by the formula

$$M^2 = \frac{1}{\alpha'} \left(\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n - 1 \right). \quad (32)$$

Let us compute the masses of the first two levels. In the vacuum state there are no contributions from the oscillators and one can see from Eq.(32) that the mass of the vacuum is negative

$$m^2 = -\frac{1}{\alpha'}, \quad (33)$$

where m^2 denotes the eigenvalue of the mass operator. Therefore, the vacuum is a tachyon. The tachyon travels faster than light and can be excited to any negative energy, thus making the theory inconsistent. The presence of tachyon shows that the chosen background of the string theory (i.e. bosonic string), if existed, is unstable.

The next states are obtained from vacuum by acting with the creation operators on $|0\rangle$

$$\alpha_{-1}^\mu |0\rangle. \quad (34)$$

Due to the commutation relations given by Eq.(17), the contribution of the oscillators is exactly +1 so that the mass of the states (34) is $m^2 = 0$. They are states in the massless representation of the group $SO(1, 25)$. Therefore, their physical degrees of freedom are in the vector representation of $SO(24)$ which is the little group of the Lorentz group in $D = 26$. If we construct the vector

$$|\xi\rangle = \xi_\mu \alpha_{-1}^\mu |0\rangle, \quad (35)$$

we can see that there are equivalent vectors to it, namely

$$|\xi'\rangle = |\xi\rangle + \lambda |\psi\rangle, \quad (36)$$

where $|\psi\rangle$ is the spurious state given in Eq.(29) and λ is an arbitrary complex number. Thus, we can interpret Eq.(36) as a $U(1)$ gauge transformation

$$\xi^\mu \rightarrow \xi^\mu + \lambda k^\mu, \quad (37)$$

where k^μ is the momentum of the spurious state. In conclusion, *the state $|\xi\rangle$ describes a massless photon in $D=26$ with 24 transverse polarizations.*

2.3 Massless spectrum of bosonic closed string

The classical string has two types of oscillation modes: left and right. The waves that propagate to the left are independent of the ones that propagate to the right. Therefore, in the quantum theory the left and right Fock spaces will be independent and the total Fock space of the closed string will be the their tensor product. Also, the operators split into operators that act on the left-modes and the ones that act on the right-modes, respectively.

Upon quantization of the solution given in Eq.(10) the Fourier modes α and $\tilde{\alpha}$ become operators acting on the right and left Fock spaces. Each of these two sets of operators

obeys the algebra (17) and the two algebras are independent [1, 2, 3]. In order to define the physical states one has to impose the vanishing of the energy-momentum tensor on the total Fock space. Moreover, since the string is closed, there is an extra symmetry that should be maintained at the quantum level, namely the world-sheet invariance under the translation along σ .

The vacuum of the theory is defined as in the case of the open string

$$|0\rangle \equiv |0\rangle_\alpha |0\rangle_{\tilde{\alpha}} |p\rangle \quad (38)$$

$$\alpha_n^\mu |0\rangle_\alpha = 0 \quad (39)$$

$$\tilde{\alpha}_n^\mu |0\rangle_{\tilde{\alpha}} = 0, \quad n > 0 \quad (40)$$

$$\hat{p}^\mu |p\rangle = p^\mu |p\rangle. \quad (41)$$

The physical states are defined by imposing half of the Virasoro operators on the Fock space as well as the invariance of the world-sheet under the translation along σ which is generated by the operator $L_0 - \tilde{L}_0$ [1, 2]

$$L_m |\phi\rangle = \tilde{L}_m |\phi\rangle = 0, \quad m > 0 \quad (42)$$

$$(L_0 - 1) |\phi\rangle = (\tilde{L}_0 - 1) |\phi\rangle = 0 \quad (43)$$

$$(L_0 - \tilde{L}_0) |\phi\rangle = 0. \quad (44)$$

The last condition above implies that the right- and left-modes should come in pairs of equal mass since the operator L_0 is related to M^2 .

The spurious states are defined through the following relations

$$(L_0 - 1) |\psi\rangle = 0 \quad (45)$$

$$(\tilde{L}_0 - 1) |\psi\rangle = 0 \quad (46)$$

$$\langle \psi | \phi \rangle = 0 \quad (47)$$

$$(L_0 - \tilde{L}_0) |\psi\rangle = 0, \quad (48)$$

where $|\psi\rangle$ is an arbitrary physical state. The last relation guarantees that the state to which a spurious state is added to remains invariant under translation by σ .

One can find the mass operator as in the open string case or by observing that the total mass should be the sum of the left and right mass operators

$$M^2 = M_L^2 + M_R^2 \quad (49)$$

$$M^2 = \frac{2}{\alpha'} \left(\sum_{n=1}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) - 2 \right). \quad (50)$$

Let us look at the first states in the spectrum. The vacuum state is a tachyon of mass

$$m^2 = -\frac{4}{\alpha'} \quad (51)$$

with all the consequences that we saw in the case of the open string. The next states are constructed by applying equal number of creation operators on vacuum from left and right sectors as dictated by the level matching condition. The first states are given by

$$\alpha_{-1}^{\mu} \tilde{\alpha}_{-1}^{\nu} |0\rangle \quad (52)$$

and it is easy to see that they are massless. In $D = 26$ there are 26×26 such of massless states. They form a tensor in a reducible representation of the group $SO(1, 25)$. It splits into irreducible representation as follows

$$\alpha_{-1}^{\mu} \tilde{\alpha}_{-1}^{\nu} |0\rangle = \begin{cases} \alpha_{-1}^{(\mu} \tilde{\alpha}_{-1}^{\nu)} |0\rangle & \rightarrow g^{\mu\nu} \\ \alpha_{-1}^{[\mu} \tilde{\alpha}_{-1}^{\nu]} |0\rangle & \rightarrow B^{\mu\nu} \\ \text{Tr } \alpha_{-1}^{\mu} \tilde{\alpha}_{-1}^{\nu} |0\rangle & \rightarrow \phi, \end{cases}$$

where $g^{\mu\nu}$, $B^{\mu\nu}$ and ϕ represent the graviton, the antisymmetric (Kalb-Ramond) field and the dilaton, respectively. The identification of the string states with the quantum fluctuations of the corresponding classical fields is justified by two arguments. The first one takes into account the equivalence under the addition of spurious states that is interpreted as gauge transformation. It is easy to see that these transformations are

$$g^{\mu\nu} \rightarrow g^{\mu\nu} + \partial^{\mu} \xi^{\nu} + \partial^{\nu} \xi^{\mu} \quad (53)$$

$$B^{\mu\nu} \rightarrow B^{\mu\nu} + \partial^{\mu} \xi^{\nu} - \partial^{\nu} \xi^{\mu} \quad (54)$$

$$\phi \rightarrow \phi + \varphi, \quad (55)$$

which are just the gauge transformation of the classical fields. The second argument relies on the interaction theory and it can be shown that the states above satisfy the correct equations of motion of the corresponding fields. Thus, we may conclude that the *massless spectrum of the closed string contains the graviton in $D=26$* .

The appearance of gravity in a natural way is one of the most attractive features of bosonic string theory. However, the theory suffers from three serious drawbacks: the presence of tachyon, the absence of fermions and the high dimensionality of space-time. The first two problems can be solved by introducing the world-sheet supersymmetry and constructing a superstring theory. It can be shown that one can obtain a space-time supersymmetric theory in the light-cone gauge which is also free of tachyon [1, 2]. In the same time the number of space-time dimensions is reduced from $D = 26$ to $D = 10$. The price to be paid is that there are now five consistent superstring theories. However, there are strong hints

that all these theories are actually different limits of an unique underlying theory unknown at present [2]. The other problem, namely reducing the spacetime dimensionality from ten to four in a natural (dynamical ?) fashion is unsolved up to day.

2.4 Exercises

Exercise 1

Starting from the bosonic string action (1) find the Neumann and Dirichlet boundary conditions using Green's theorem in two dimensions:

$$\int_{\Sigma} dx dy (\partial_x Q - \partial_y P) = \int_{\partial \Sigma} P dx + Q dy. \quad (56)$$

Exercise 2

Prove any of the solutions of the open string with N-N, D-D, N-D and D-N boundary conditions and the closed string solution by using the method of separation of variables.

Exercise 3

Starting from equal-time commutator (15) show that the operators α satisfy the algebra (17) and that the coordinates and the momenta of the center of mass of string satisfy the relation (18).

Exercise 4

Show that the total momentum of the open string defined as

$$P_{total}^{\mu} = \int_C d\sigma^1 P_0^{\mu} + d\sigma^0 P_1^{\mu}, \quad (57)$$

where C is an arbitrary curve on the world-sheet from the boundary $\sigma = 0$ to $\sigma = \pi$, is conserved.

3 Bosonic D -branes. Effective Action

In this lecture I am going to introduce the D -branes and to discuss their physical degrees of freedom following [2, 4]. Also, I will present the way in which the background field method is applied in order to obtain the Born-Infeld effective action of D -branes (see [2, 4, 6]).

3.1 Definition of bosonic D -branes

By definition, a bosonic D -brane is an extended physical object on which bosonic open strings can end [2, 4].

As we saw in the previous lecture, the momentum of the open string should be conserved and it is reasonable to assume that the momentum does not flow away through the ends of the string. However, when the open string ends on a D -brane, the situation is different. Indeed, there can be an exchange of momentum between the string and the brane through the end of string that is in contact with the brane. Therefore, it is the momentum of the full system that should be conserved. Another immediate consequence of introducing branes in string theory is that the Lorentz invariance is now broken.

The word “physical” in the definition means that the branes are characterized by more than their geometry and topology and that they have some physical properties like tension and charge as we shall see in the next lectures. The real motivation for introducing the D -branes was that in the spectrum of Type I and Type II superstring theories there are some bosonic fields which are described by p -forms in $D = 10$ dimensions. At that time it was not known what could have been the sources of such fields and it was discovered by Polchinsky that the D -branes were the sought for objects [7].

It follows from the definition of the D -branes that the open strings that end on them must have Dirichlet boundary conditions on the directions transversal to the branes and Neumann boundary conditions on the directions tangent to the brane. Indeed, there is nothing that can stop the string of sliding on the world-volume of the brane. In the simplest case a Dp -brane is a hypersurface embedded in the $D = 10$ space-time where p indicates that it has p space-like directions. If the D -brane is situated at the $\sigma = 0$ end of string then the boundary conditions are given by the following relations

$$\text{N.b.c. } \partial_\sigma X^a|_{\sigma=0} = 0, \quad a = 0, 1, 2, \dots, p, \quad (58)$$

$$\text{D. b. c. } X^i|_{\sigma=0} = x^i, \quad i = p + 1, \dots, 9. \quad (59)$$

Actually, by computing the spectrum of the open strings ending on the branes we can find the degrees of freedom of the brane, that is the fields that live on the world-volume. We are interested in the massless degrees of freedom which will not change the energy of the brane. They are given by the strings with both ends on the brane. The other strings will contribute with an energy proportional to the stretching of the string. For example, the strings between two branes will contribute with the following stretching energy [2, 4]

$$m^2 = \frac{Y^2}{4\pi^2\alpha'}. \quad (60)$$

The relation above can be obtained by considering the T-duality of the theory [2, 4] but it can also be established from dimensional arguments.

In order to find the massless degrees of freedom we take the solution of the string equations of motion with Dirichlet boundary conditions at the two ends in the transverse

directions (12) and Neumann in the tangential directions (11). After quantizing them as in the previous section we discover that the massless states are given by

$$\begin{aligned}\alpha_{-1}^a|0\rangle &, \quad a = 0, 1, \dots, p \\ \alpha_{-1}^i|0\rangle &, \quad i = p+1, \dots, 25.\end{aligned}\tag{61}$$

The first set of states describes an $SO(1, p)$ photon $A^a(\xi)$ while the second one is associated to an $SO(25 - p)$ massless vector $\Phi^i(\xi)$. The components of the latter are associated to the breaking of the translational symmetry along the transverse directions X^i and are interpreted as the fluctuations around the classical localization of brane in the transverse spacetime. By ξ we denoted the coordinates on the world-volume of the D -brane. Actually, the fields $\Phi^i(\xi)$ represent a particular embedding of the D -brane in spacetime. In general, the corresponding degrees of freedom are the embedding functions $X^\mu(\xi)$ of the world-sheet volume in the target space. In the above case we have considered the simplest situation in which the brane was flat and its tangential directions were parallel to some of the directions of spacetime.

Thus, a Dp -brane breaks the space-time symmetry of the theory as follows

$$SO(1, 25) \rightarrow SO(1, p) \times SO(25 - p)\tag{62}$$

and consequently its massless degrees of freedom are given by the set $\{A^a(\xi), X^\mu(\xi)\}$.

3.2 Effective action of Dp -branes

It is possible now to find the dynamics of the Dp -brane in the low energy limit. Indeed, in this limit the degrees of freedom of the brane are the classical fields found in the previous section. One should look for an action describing the dynamics of these fields and we end up with the effective field theory of the brane.

Recall that the degrees of freedom of the branes were found in terms of open strings ending on them. In order to have a description consistent with two dimensional string theory, one should stick on the conformal invariance of string in the new background in which the Dp -branes are present ([2, 4]). Without the conformal invariance the two dimensional theory will not describe a physical theory. This requirement is the same for strings in *any arbitrary* background and it is implemented as follows. The string theory in a general background contains couplings between strings and the background fields. However, these coupling terms break in general the conformal invariance. In order to find those configurations which preserve the two dimensional conformal invariance, one treats the background fields as coupling constants and the sought for configurations can be found by solving the

equation

$$\beta_\Gamma = 0, \quad (63)$$

where β_Γ is the beta-function of any background field Γ [1, 2]. One way to compute the beta-functions is by using the *background field method* (see [8, 9].) We are going to show how this method is applied to obtain the low energy action of Dp -branes following ([10]).

In a background containing Dp -branes the open strings couple with the brane degrees of freedom $\{A^a(\xi), X^\mu(\xi)\}$. Beside them, there may be other massless fields in the background like, for example, the closed string fields $g_{\mu\nu}, B_{\mu\nu}, \phi$. If the theory is supersymmetric, then massless fermions are also present. Each of these background fields will have a beta-function that must vanish if the two-dimensional theory that describes strings is to be conformal.

D25-brane in a flat background

To understand how the beta-functions are computed, let us consider firstly a simpler situation in which we have a $D25$ -brane that fills the whole space-time and no closed string fields in the background. The only background fields are $A^\mu(X)$. The photon couples with one dimensional world-line or, equivalently, with dimensionless charges like in electrodynamics. Therefore, in order to couple it with the string which has a two-dimensional world-sheet, we have to put some “electric charges” at the ends of the open strings (which are just points) and to couple the photon with these charges in the usual way. Actually, this explains the presence of the $U(1)$ field on the world-volume of the brane as being generated by the charges at the end of the open string. Let us consider both the space-time and the world-sheet Euclidean and map the world-sheet to the complex upper-half plane with $z = \tau + i\sigma$. The $U(1)$ field couples on the boundary of the world sheet and the total action is given by the formula

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2z \partial_\alpha X^\mu \partial^\alpha X_\mu + \frac{i}{2\pi\alpha'} \int_{\partial\Sigma} d\tau A_\mu(X) \dot{X}^\mu, \quad (64)$$

where A_μ has been rescaled to include a $2\pi\alpha'$ factor and the $U(1)$ charge has been taken 1. Choose a background field $\bar{X}^\mu(\tau, \sigma)$ that is a solution of the equations of motion and of boundary conditions that are derived from Eq.(64) above. Now expand the fields $X(\tau, \sigma)$ around this solution. One obtains the following set of equations

$$\begin{aligned} X^\mu(\tau, \sigma) &= \bar{X}^\mu(\tau, \sigma) + \zeta^\mu(\tau, \sigma) \\ \square \bar{X}^\mu(\tau, \sigma) &= 0 \\ \partial_\sigma \bar{X}^\mu + iF_\nu^\mu \partial_\tau \bar{X}^\nu|_{\partial\Sigma} &= 0, \end{aligned} \quad (65)$$

where $\square = \partial_\sigma^2 + \partial_\tau^2$ is the Laplacean on the Euclidean world-sheet, $F_{\mu\nu} = \nabla_{[\mu} A_{\nu]}$ and $\nabla_\mu = \partial/\partial X^\mu$. The full information on the field is contained in the fluctuation ζ around

the background \bar{X} [8, 9, 11]. Introducing (65) in the action (64) we obtain the expansion around the background solution. We consider only slow varying fields $F_{\mu\nu}$. This condition allows us to disregard higher derivatives of F like $\nabla^2 F$, $\nabla^3 F$, \dots . The expanded action takes the form

$$\begin{aligned} S[\bar{X} + \zeta] &= S[\bar{X}] + \frac{1}{2\pi\alpha'} \int_{\Sigma} d^2z (\partial_{\alpha} \bar{X}^{\mu} \partial^{\alpha} \zeta_{\mu} + \frac{1}{2} \partial_{\alpha} \zeta^{\mu} \partial^{\alpha} \zeta_{\mu} + \dots) \\ &+ \frac{i}{2\pi\alpha'} \int_{\partial\Sigma} d\tau (F_{\mu\nu} \zeta^{\mu} \partial_{\tau} \bar{X}^{\nu} + \frac{1}{2} \nabla_{\rho} F_{\mu\nu} \zeta^{\rho} \zeta^{\nu} \partial_{\tau} \bar{X}^{\mu} \\ &+ \frac{1}{2} F_{\mu\nu} \zeta^{\mu} \partial_{\tau} \zeta^{\nu} + \frac{1}{3} \nabla_{\rho} F_{\mu\nu} \zeta^{\rho} \zeta^{\nu} \partial_{\tau} \zeta^{\mu} + \dots). \end{aligned} \quad (66)$$

We look for the one loop beta-function. This is given by the one-loop counterterm with one external leg $\partial_{\tau} \bar{X}$ of the interaction term in (64), that is

$$i \int_{\partial\sigma} d\tau A_{\mu} \partial_{\tau} \bar{X}^{\mu} \rightarrow \Delta S_I[\bar{X}] = \frac{i}{2\pi\alpha'} \int_{\partial\Sigma} d\tau \Gamma_{\mu} \bar{X}^{\mu}. \quad (67)$$

The value of the corresponding one-loop Feynman diagram is

$$\Delta S_I[\bar{X}] = -\frac{i}{2\pi\alpha'} \int d\tau \frac{1}{2} \nabla_{\rho} F_{\mu\nu} \partial_{\tau} \bar{X}^{\mu} G^{\rho\nu}(\tau, \tau')|_{\tau \rightarrow \tau'}, \quad (68)$$

where G is the Green's function computed on the boundary $\sigma = 0$ in the point τ . Thus, it is the solution of the following problem

$$\frac{1}{2\pi\alpha'} \square G_{\mu\nu}(z, z') = -\delta_{\mu\nu} \delta(z - z') \quad (69)$$

$$\partial_{\sigma} G_{\mu\nu} + i F_{\mu}^{\lambda} \partial_{\tau} G_{\nu\lambda}|_{\partial\Sigma} = 0. \quad (70)$$

One can find the explicit form of the Green's function by using the method of images [11] and it is given by the following relation

$$G_{\mu\nu} = \alpha' [\delta_{\mu\nu} \ln |z - z'| + \frac{1}{2} (\frac{1-F}{1+F})_{\mu\nu} \ln(z - \bar{z}') + \frac{1}{2} (\frac{1+F}{1-F})_{\mu\nu} \ln(\bar{z} - z')], \quad (71)$$

where we have used the notation

$$(\frac{A}{B})_{\mu\nu} = A_{\mu}^{\rho} (B^{-1})_{\rho\nu}. \quad (72)$$

When $F = 0$ Eq.(71) reduces to the known Green's function in the absence of $U(1)$ field. The Green's function on the boundary is

$$G_{\mu\nu}(\tau \rightarrow \tau') = -2\alpha' \ln \Lambda (1 - F)_{\mu\nu}^{-1}, \quad (73)$$

where Λ is a short distance cut-off. The beta-function of the field A^{μ} is given by applying the definition and it should vanish in order to have a conformal invariant theory

$$\beta_{\mu}^A = \Lambda \frac{\partial \Gamma_{\mu}}{\partial \Lambda} = \nabla^{\rho} F_{\mu}^{\lambda} (1 - F^2)_{\lambda\rho}^{-1} = 0. \quad (74)$$

The effective action is the action from which the Eq.(74) can be defined through the variational principle, i. e. the action which has the equations of motion given by Eq.(74). Actually, there is no such of action [11] and therefore one has to find an equation that is equivalent to (74). Such of equation is

$$\chi^{\mu\nu}(F)\beta_\nu^A = 0, \quad (75)$$

for any invertible matrix $\chi^{\mu\nu}(F)$. Now after some algebra [10], one can show that the sought for equation of motion is

$$\sqrt{\det(1+F)}(1-F^2)^{-1}_{\mu\nu}\beta_A^\nu = 0, \quad (76)$$

which can be derived from a non-polynomial action called *Born-Infeld action* given by the integral of

$$L_{BI} = \exp\left[\frac{1}{2}\text{Tr}\ln(1+F)\right] = [\det(1+F)]^{\frac{1}{2}}. \quad (77)$$

The above Lagrangian describes the effective action of the massless states of the open string in a background that contains an $U(1)$ gauge potential that couples with the boundary of the world-sheet. According to D-brane interpretation this is the effective action of a $D25$ -brane.

D25-brane in closed string background

The situation can be complicated further to include other fields in background. Since all the fields should come from the string spectrum, we may include other massless or massive string fields. Let us consider a background in which the graviton, the dilaton and the Kalb-Ramond two-form field do not vanish. The action for the open string contains the following terms

$$S = S_g + S_B + S_\phi + S_A \quad (78)$$

where

$$\begin{aligned} S_g &= \frac{1}{4\pi\alpha'} \int_\Sigma d^2z g_{\mu\nu}(X) \partial_\alpha X^\mu \partial^\alpha X^\nu \\ S_B &= -\frac{i}{4\pi\alpha'} \int_\Sigma d^2z \epsilon^{\alpha\beta} B_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \\ S_\phi &= \frac{1}{4\pi\alpha'} \int_\Sigma d^2z \left(-\frac{1}{2}\alpha'\right) \sqrt{h} R^{(2)} \phi(X) + \frac{1}{2\pi\alpha'} \int_{\partial\Sigma} d\tau \left(-\frac{1}{2}\alpha'\right) k \phi(X) \\ S_A &= \frac{i}{2\pi\alpha'} \int_{\partial\Sigma} d^2z A_\mu(X) \partial_\tau X^\mu, \end{aligned} \quad (79)$$

where $\epsilon_{\alpha\beta}$ is the two-dimensional antisymmetric symbol, h is the determinant of the two-dimensional metric $h_{\alpha\beta}$ which at the tree-level is Minkowski, at one-loop is cylindric, etc. and $R^{(2)}$ is the two-dimensional curvature.

The background field A^μ is treated as a coupling constant as in the previous example. In order to have a conformal invariant field theory in two-dimensions, the beta-function of it should vanish. The beta-function can be computed using the same method as above. The field X^μ is chosen to satisfy the free field theory with interaction with the Kalb-Ramond and $U(1)$ gauge-potential on the boundary. The world-sheet is the same as in the previous case. The equations of motion and the boundary conditions that are obtained from the sigma-model (78) are given by the following relations ([9, 11])

$$[g_\nu^\mu \partial_\alpha + \Gamma_{\nu\lambda}^\mu \partial_\alpha \bar{X}^\lambda + \frac{i}{2} H_{\nu\lambda}^\mu \epsilon^{\alpha\beta} \partial_\alpha \bar{X}^\lambda] \partial_\alpha \bar{X}^\nu = 0 \quad (80)$$

$$\partial_n \bar{X}^\mu - i(B + F)_\nu^\mu \partial_\tau \bar{X}^\nu|_{\sigma=0} = 0, \quad (81)$$

where

$$H_{\mu\nu\rho} = 3\nabla_{[\rho} B_{\mu\nu]}. \quad (82)$$

The contribution to the compensating term, at one-loop, gives the following beta-function of A^μ [11]

$$\begin{aligned} \beta_\mu^A &= \nabla^\rho (B + F)_\nu^\mu [g - (B + F)^2]_{\nu\rho}^{-1} \\ &+ \frac{1}{2} (B + F)_{\mu\nu} H^{\nu\lambda\rho} \left[\frac{B + F}{g - (B + F)^2} \right]_{\lambda\rho} + \frac{1}{2} \nabla^\nu \phi (B + F)_{\mu\nu} \end{aligned} \quad (83)$$

The invertible matrix that generates a variational equation from the equations that imposes the conformal invariance on the system is

$$\chi_{\mu\nu} = (g - (B + F)^2)_{\mu\nu}^{-1}. \quad (84)$$

The Born-Infeld effective action is given by the following relation

$$S_{eff} \simeq \int d^{26} X e^{\frac{1}{2}\phi} [\det (g + B + F)]^{1/2} \quad (85)$$

from which one obtains the following equation of motion

$$e^{-\frac{1}{2}} [\det(g + B + F)]^{\frac{1}{2}} (g - (B + F)^2)_{\mu\nu}^{-1} \beta_A^\nu = 0. \quad (86)$$

The \simeq means that the action is given up to some dimensional constant. This constant is necessary in order to make the left hand side of (85) an action and from dimensional arguments one sees that it should have the dimension of the brane tension T .

Dp -brane effective action

The Born-Infeld action of the low energy effective field theory of a generic Dp -brane can be obtained as in the two examples above. Actually, one can easily adapt the action

(85) to serve this purpose. To this end, note that in the case of the Dp -branes the fields will interact with the $(p+1)$ -dimensional world-volume. As was discussed at the beginning of this section, A^μ are fields living on the world-volume, therefore they will depend on the world-volume coordinates ξ . The rest of the fields live in the full space-time, but they interact with the world-volume through some “world-volume projected” components. This projection is given by the pull-back of the embedding $X^\mu(\xi)$ of the world-volume into the space-time. We denote by $\hat{}$ these fields. Then the Born-Infeld action is given by

$$S_{Dp} = -T_p \int \delta^{p+1} \xi e^{-\hat{\phi}} [\hat{g}_{ab} + \hat{B}_{ab} + 2\pi\alpha' F_{ab}]^{1/2}, \quad (87)$$

where T_p is the tension of the brane, the pull-back of the fields are

$$\begin{aligned} \hat{g}_{ab} &= \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} \\ \hat{B}_{ab} &= \partial_a X^\mu \partial_b X^\nu B_{\mu\nu} \\ \hat{\phi} &= \phi(\xi), \end{aligned} \quad (88)$$

and ∂_a denotes the world-volume derivative $\partial/\partial\xi^a$. The field-strength of the $U(1)$ form is a world-volume field and therefore is not pulled-back. The dilaton coupling is -1 since we consider the coupling with the disk.

The action (87) has two gauge symmetries, a $U(1)$ one and a Kalb-Ramond one, given by the following transformation laws

$$\delta A = d\lambda \ ; \ \delta B = 0 \quad (89)$$

$$\delta B = \delta\zeta \ ; \ \delta A = -\frac{1}{2\pi\alpha'}\zeta. \quad (90)$$

Only the combination

$$\hat{B}_{ab} + 2\pi\alpha' F_{ab} \quad (91)$$

is invariant under both gauge transformations, which explains the presence of the pull-back of the Kalb-Ramond field in the action, even if it does not couple directly with the world-volume of the Dp -brane. The factor

$$\epsilon^{-\phi} = g_s^{-1} \quad (92)$$

is proportional with the inverse of the string coupling. Therefore, by varying the dilaton expectation value, one can study the dynamics of D -branes in different regimes. This situation is familiar from string theory in which the coupling constant is dynamical. If we take $F_{ab} = \hat{B}_{ab} = \phi = 0$ then the action is proportional to the geometric volume of the world-volume

$$S = -T_p \int d^{p+1} \xi \sqrt{\hat{g}}. \quad (93)$$

3.3 Exercises

Exercise 1

Calculate the expansion in (66).

Exercise 2

Find the Green's function for the two dimensional Laplace operator from

$$S = \frac{g}{4\pi} \int d^2z \partial \Phi \bar{\partial} \Phi \quad (94)$$

and put the appropriate boundary conditions.

Exercise 3

Find the Green's function for the following problem

$$\begin{aligned} \frac{1}{2\pi\alpha'} \square G(z, z') &= -\delta(z - z') \\ \partial_\sigma G(z, z')|_{\partial\Sigma} &= 0 \end{aligned} \quad (95)$$

on the upper half-plane ($z = \tau + i\sigma$).

Exercise 4

Using the Fourier transformation of the Green's function in the upper half-plane

$$G(z, z') = \int \frac{dp}{2\pi} \frac{e^{ip(\tau - \tau')}}{2|p|} [e^{-|p||\sigma - \sigma'|} + e^{-|p|(\sigma + \sigma')}] \quad (96)$$

find $G_{\mu\nu}(\tau \rightarrow \tau')$ on the boundary.

Exercise 5

Prove the following identity

$$(1 - F^2)^{-1}_{\mu\nu} \beta_A^\nu = \nabla^\nu \left(\frac{F}{1 - F^2} \right)_{\mu\nu} - \left(\frac{F}{1 - F^2} \right)_{\mu\lambda} \nabla^\nu F^{\lambda\rho} \left(\frac{F}{1 - F^2} \right)_{\rho\nu}, \quad (97)$$

where

$$\beta_A^\nu = \nabla^\nu F_\mu^\lambda (1 - F^2)^{-1}_{\lambda\nu}. \quad (98)$$

4 Bosonic D-branes. Tension

In this section we are going to compute the tension of the D-branes by computing the interaction amplitude in the string theory and then comparing it with the field theory computations.

4.1 String computation

The D-branes interact by exchanging closed strings in various quantum states in analogy with the interaction between particles that exchange some other (virtual) particles. There is some response in the brane to the exchange of closed string excitations and this response should be proportional to the tension of the brane. The quantity that measure the intensity of the exchange of closed string states is the exchange amplitude.

We are going to do this computation at the tree level in perturbation string theory because we want to compare latter the result with the corresponding calculations in the low energy limit field theory. Since in this limit only massless quanta participate to the interaction, we have to take into account only the effects produced by these string modes.

One way to do these computations is to interpret the tree level Feynman diagram for closed strings as one-loop diagram for open strings. Let us see how this is done. At tree level, a closed string emitted at the moment $\tau = 0$ propagates along the cylinder an interval T . Therefore, the horizontal coordinate of the cylinder is $0 \leq \tau \leq T$ and the periodic one is $0 \leq \sigma \leq \pi$, the space-like parameter of the closed string. However, the same cylinder can be interpreted as an open string of length $0 \leq \sigma \leq \pi$ that propagates on a loop in the time $0 \leq \tau \leq \hat{T}$. In this case the horizontal coordinate of the cylinder is parametrized by σ . Then the two amplitudes in closed string and open string description (called also *channels*) should be equal. To have the same cylinder in the two cases, the parameters (τ, σ) of the closed and open strings should be adjusted in such of way that the interval T parametrized by τ in closed string channel be equal to π parametrized by σ in open string channel, which gives $\hat{T} = \pi/T$.

The amplitude in the open string channel

The one-loop vacuum amplitude in QED is given by the logarithm of the partition function

$$\mathcal{A} = \ln(Z_{vac}) \tag{99}$$

and it can be calculated by using the Coleman-Weinberg formula that can be obtained as follows [4]. Start with the logarithm of the partition function for a scalar field given by the following relation [2, 4]

$$\ln(Z_{vac}) = -\frac{1}{2}\text{Tr} \ln(\square + m^2) = -\frac{V_d}{2} \int \frac{d^d k}{(2\pi)^d} \text{Tr} \ln(k^2 + m^2), \tag{100}$$

where d is the number of the dimensions of space-time, V_d is the volume in which the field is contained and $(k^2 + m^2)/2 = H$ is the Hamiltonian of the field. Then use the following

following property of the \ln function

$$\ln x = - \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{\infty} \frac{dt}{t} e^{-tx}. \quad (101)$$

By inserting (101) into (100) one obtains the Coleman-Weinberg formula for a scalar field

$$\mathcal{A} = V_d \int \frac{d^d k}{(2\pi)^d} \int_0^{\infty} \frac{dt}{2t} \text{Tr} e^{-(k^2+m^2)t/2}. \quad (102)$$

When one integrates on the circle, the two orientation of it are taken into account, that is why we have to divide the integrand of (102) by a factor of 2. The relation (102) has the interpretation of the *free energy*.

We are going to apply now the formula (102) to the modes of the open strings that move on the circle parametrized by $2\pi\tau$ at one-loop. We know from the second section that the Hamiltonian of the open string is given by

$$H = L_0 - 1 = \alpha' (k^2 + \hat{M}^2) \quad (103)$$

where the Virasoro operator L_0 is given by the relation

$$L_0 = \alpha' k^2 + \alpha' \frac{Y^2}{(2\pi\alpha')^2} + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n. \quad (104)$$

The term proportional to the distance Y^2 between the branes is due to the stretching energy of the string. Then the amplitude for the open string modes can be computed by applying directly the formula (102). This is possible since the string can be viewed as a collection of scalar fields in two-dimensions. Then the sought for amplitude is given by the following relation

$$\mathcal{A} = \int_0^{\infty} \frac{d\tau}{2\tau} \text{Tr} e^{-2\pi\tau(L_0-1)}. \quad (105)$$

or, by plugging (104) into (105)

$$\mathcal{A} = \int_0^{\infty} \frac{d\tau}{\tau} \times V_{p+1} \int \frac{d^{p+1}k}{(2\pi)^{p+1}} e^{-2\pi\tau\alpha'k^2} e^{-\frac{Y^2\tau}{2\pi\alpha'}} e^{2\pi\tau} \text{Tr} \left[e^{-2\pi\tau \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n} \right]. \quad (106)$$

The factor of 2 disappeared from the denominator since we allow the interchange of the orientation of open string and each orientation gives an equal contribution to the amplitude. Note in (105) the expression of the mass-shell condition $(\square+m^2)\phi=0 \leftrightarrow (L_0-1)|\phi\rangle=0$ in the QFT and string theory, respectively. From the last relation we will write the propagator of the string later.

To compute the r.h.s. of (106) we note that it factorizes into an integral over k and the trace over the oscillation modes. The integral is Gaussian and from it we will obtain the factor

$$(8\pi^2\alpha'\tau)^{-\frac{p+1}{2}}. \quad (107)$$

Also, by computing the trace of \hat{k}'^2 in the parallel directions to the world-volume of the brane, the volume V_{p+1} of the brane is obtained. We use the following normalization relation

$$\begin{aligned} \langle k|k' \rangle &= 2\pi\delta(k - k') \\ V_{p+1} &= (2\pi)^{p+1}\delta^{p+1}(0). \end{aligned} \quad (108)$$

The trace over the oscillators can be computed in the basis of the operators α_{-n} and α_n

$$\begin{aligned} \text{Tr} e^{-2\pi\tau \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n} &= \prod_{n=1}^{\infty} \prod_{\mu=0}^{25} \text{Tr} e^{-2\pi\tau \alpha_{-n}^{\mu} \alpha_{n\mu}} = \prod_{n=1}^{\infty} \prod_{\mu=0}^{25} \sum_{m=0}^{\infty} \langle m|e^{-2\pi\tau n a_{-n}^{\dagger} a_n}|m \rangle \\ &= \prod_{n=1}^{\infty} \left(\frac{1}{1 - e^{-2\pi\tau n}} \right)^{26} \end{aligned} \quad (109)$$

where $a_n^{\dagger} a_n |m \rangle = m |m \rangle$. The trace above includes the contribution of the non-physical degrees of freedom. To remove them, one should pick-up a gauge. In any covariant gauge, the non-physical degrees of freedom are taken account of by the Fadeev-Popov ghosts. Without writing their explicit contribution we give the final form of the amplitude

$$\mathcal{A} = V_{p+1} (8\pi^2 \alpha')^{-\frac{p+1}{2}} \int_0^{\infty} \frac{d\tau}{\tau} \tau^{-\frac{p+1}{2}} e^{-\frac{Y^2 \tau}{2\pi\alpha'}} [f_1(e^{-\pi\tau})]^{-24}. \quad (110)$$

We see that the effect of the ghosts is to reduce the number of space-time dimensions by two, i.e. to the transverse directions. This can be also done by solving firstly the constraints, which will leave the theory in the light-cone gauge [1]. The function f_1 is defined as

$$f_1(q) = q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^{2n}) \quad (111)$$

and under a *modular transformation* of its variable

$$\tau \longrightarrow \frac{1}{\tau} \quad (112)$$

it transforms in the following way

$$f_1(e^{-\pi/\tau}) = \sqrt{\tau} f_1(e^{-\pi\tau}) \quad (113)$$

which is the modular transformation property of the function $f_1(q)$.

The amplitude of closed string massless modes

Now we would like to identify in the amplitude (110) the contribution of the closed string modes which interest us. To this end, we note that in the limit $\tau \rightarrow \infty$, i.e. when

the circle of the cylinder opens, the world-sheet becomes a long and thin strip. In order for a mode of the open string to travel the loop, it should be light since it has to reach across a long distance. Thus, in this limit the light modes of the open string dominate the amplitude. In the limit $\tau \rightarrow 0$, the open string is in the UV regime since the radius of the circle is small and the string modes have to travel short distances in making the loop. However, this limit is the long-distance of the closed string. Indeed, by making a reparametrization of the string length (a conformal transformation) that does not change the area of the cylinder while it makes its radius small, we see that the length of the cylinder goes as

$$Y_1 = \frac{Y_0}{2\epsilon} \longrightarrow \infty, \quad (114)$$

for any scale unit $\epsilon = \tau \rightarrow 0$, where Y_0 is the distance between branes. Y_1 is the apparent length of the cylinder as viewed by the string modes. In the closed string channel, the closed string modes have to travel this distance between branes and therefore this is the UV limit of closed strings in which its light modes have a major contribution.

All we have to do now is to make the modular transformation of the cylinder parameter τ given by (112) and to make the expansion of f_1 function in $\tau \rightarrow 0$ limit

$$[f_1(e^{-\pi x})]_{x \rightarrow \infty}^{24} = \sum_{n=0} c_n e^{-2\pi x(n-1)} = e^{2\pi x} + 24 + O(e^{-2\pi x}). \quad (115)$$

Each term in the power expansion corresponds to the trace of closed string modes with mass

$$\frac{\alpha' M^2}{2} = 2(n-1). \quad (116)$$

The first term is the contribution of the tachyon and we are only interested in the second term which represents the contribution of the closed string massless modes. The sought for interaction amplitude is given by the following relation

$$\mathcal{A} = V_{p+1} \frac{24\pi}{2^{10}} (4\pi^2 \alpha')^{11-p} G_{25-p}(Y) \quad (117)$$

where the Green's function in the transverse directions to the world-volume of the brane is given by the relation

$$G_{25-p}(Y) = 2^{-2} \pi^{\frac{25-p}{2}} \Gamma\left(\frac{1}{2}(25-p) - 1\right) Y^{2+p-25}. \quad (118)$$

Here, $\Gamma\left(\frac{1}{2}(25-p) - 1\right)$ is the Gamma-function.

4.2 Field theory computations

The amplitude that has been obtained in (117) describes the interaction of the Dp -branes via the exchange of closed string massless modes. We saw in the second section that these

modes are identified with the quanta of the gravitational, dilaton and Kalb-Ramond fields. We saw in the previous section that the low energy effective field theory that describes the Dp -brane dynamics is given by a Born-Infeld action. This action was obtained by requiring the conformal invariance of the open strings in the background that contains a Dp -brane.

This idea can be applied to string field in any background. The result will be, as in the case of the Dp -brane, an effective action that describes the dynamics of the background fields. Not all background fields will preserve the conformal invariance of the string theory but only those ones that satisfy the vanishing beta-function condition which is identified with the equations of motion of the classical fields.

In a general background that contains only ϕ , $G_{\mu\nu}$ and $B_{\mu\nu}$ fields, the effective action of them is given by the following σ -model action

$$S = \frac{1}{2k_0^2} \int d^{26}X \sqrt{-G} e^{-2\phi} \left[R + 4D_\mu \phi D^\mu \phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right], \quad (119)$$

where H is the field-strength of B

$$H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]}$$

and D_μ is the space-time covariant derivative [1, 2]. The action (119) can be obtained by using the background field method exposed in the previous section. Its equations of motion are equivalent to the conformal condition

$$\beta^\phi = \beta_{\mu\nu}^G = \beta_{\mu\nu}^B = 0, \quad (120)$$

where all the fields are treated as coupling constants and open string fields are absent from the background.

The action (119) describes the dynamics of the fields we are interested in in the bulk. However, before computing the interaction amplitude from it, we would like to decouple the dilaton from the curvature in order to benefit from the results of general relativity. The effective action in (119) is known as being in the *string frame*, and we want to write it in the *Einstein frame* by making the following rescaling of the metric

$$g_{\mu\nu} = e^{\frac{\phi_0 - \phi}{6}} G_{\mu\nu}, \quad (121)$$

where ϕ_0 is the v.e.v. of the dilaton.

In order to find the quantum amplitude we go to the linearized form of the action (119). To this end, we expand the the background field around their classical values

$$\begin{aligned} \phi &= \phi_0 + \kappa_0 \\ g_{\mu\nu} &= \eta_{\mu\nu} + \kappa_0 h_{\mu\nu} \end{aligned} \quad (122)$$

where k_0 is the gravitational coupling constant and the expansion parameter. Then we set all the v.e.v. to zero with the exception of $\eta_{\mu\nu}$.

In order to construct the Feynman diagrams that describe the interaction between fields and branes we must know the coupling constants between these fields and the Dp -branes. The coupling constant are given by the interaction term which is the linearized form of the effective action of the brane. We recall that in the string frame it is given by the relation

$$S_{BI} = -T_p \int d^{p+1} \xi e^\phi \left[-\det(\hat{G}_{ab} + \hat{B}_{ab} + 2\pi\alpha' F_{ab}) \right]^{1/2}. \quad (123)$$

To the leading order in the the gravitational coupling, the interaction between fields and branes has the form

$$\left[-\det(\eta_{ab} + k_0(h_{ab} + Bab + 2\pi\alpha' F_{ab})) \right]^{1/2} = \frac{3}{2} + \frac{1}{2}\kappa_0 h_a^a + O(\kappa_0^2). \quad (124)$$

We see that the antisymmetric tensors decouple and the only contribution is from dilaton and graviton. Then the linearized action of the fields that interact with the Dp -brane, in the Einstein frame, have the following form

$$S_{int} = \frac{1}{2k_0^2} \int d^{26} X \sqrt{-g} \left(R - \frac{1}{6} D_\mu \phi D^\mu \phi \right) \quad (125)$$

for the bulk action and

$$\begin{aligned} S_{BI \text{ int}} &= -\frac{T_p}{\kappa_0} \int d^{p+1} \xi e^{\frac{p-11}{2}\phi} \sqrt{-\det \hat{g}_{ab}}. \\ &= S_{cl} - T_p \int d^{p+1} \xi \left(\frac{p-11}{2} \phi + \frac{1}{2} h_a^a \right) + O(\kappa_0), \end{aligned} \quad (126)$$

where S_{cl} is the classical action. Since we are interested in the classical effects we are going to compute the tree level amplitude. The brane acts as sources of fields which propagate from one brane to the other. Now let us compute the propagators.

The linearized part of the graviton interaction is given by the following Lagrangian

$$\mathcal{L}_{int} = -\frac{1}{2} \partial_\lambda h_\mu^\lambda \partial^\mu h_\nu^\nu + \frac{1}{2} \partial_\lambda h_\mu^\lambda \partial_\nu h^{\nu\mu} - \frac{1}{4} \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} + \frac{1}{4} \partial_\lambda h_\mu^\mu \partial^\lambda h_\nu^\nu \quad (127)$$

with the gauge gauge invariance

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \quad (128)$$

Since there are gauge degrees of freedom, one has to fix the gauge by adding a gauge breaking term which can be chosen to be

$$\mathcal{L}_c = -\frac{1}{2} C^2, \quad (129)$$

where

$$C^\mu = \partial_\nu h^\mu{}_\nu - \frac{1}{2} \partial^\mu h^\nu{}_\nu. \quad (130)$$

The gauge fixed Lagrangian is the sum between (127) and (129)

$$\mathcal{L}_{int \text{ gauge fixed}} = \mathcal{L}_{int} + \mathcal{L}_c = -\frac{1}{2} [\partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} - \frac{1}{2} \partial_\lambda h_\mu{}^\mu \partial^\lambda h_\nu{}^\nu]. \quad (131)$$

By integrating by parts to put in evidence the propagator and adding the dilaton part we obtain the following relation

$$S_{int} = -\frac{1}{2\kappa_0} \int d^{26}X \left\{ \frac{1}{2} h_{\mu\nu} \left[\eta^{\mu\lambda} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\lambda} - \frac{1}{12} \eta^{\mu\nu} \eta^{\lambda\sigma} \right] \partial^2 h_{\lambda\sigma} + \frac{1}{6} \phi \partial^2 \phi \right\}. \quad (132)$$

By definition, the propagators are given by the functional derivatives of the action with respect to the fields

$$D_{\mu\nu, \lambda\sigma} = -\frac{\delta^2 S_{int}}{\delta h^{\mu\nu} \delta h^{\lambda\sigma}} \Big|_{h_{\mu\nu}=0} \quad (133)$$

$$D = -\frac{\delta^2 S_{int}}{\delta \phi \delta h^{\phi\phi}} \Big|_{\phi=0} \quad (134)$$

It is easy to write down the propagators for the graviton and the dilaton

$$D_{\mu\nu, \lambda\sigma}(\kappa) = -2\kappa_0^2 \left(\eta^{\mu\lambda} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\lambda} - \frac{1}{12} \eta^{\mu\nu} \eta^{\lambda\sigma} \right) \frac{1}{\kappa^2}, \quad (135)$$

$$D(\kappa) = -6\kappa_0^2 \frac{1}{\kappa^2}. \quad (136)$$

The currents necessary to write down the values of the Feynman diagrams can be read off the action linearized action (126) and are simply the coefficients of the fields

$$j_\phi = \frac{p-11}{12} T_p \delta_\perp \quad (137)$$

$$T_{\mu\nu} = \frac{1}{2} T_p \delta_\perp \times \begin{cases} \eta_{\mu\nu} & \text{for } \mu, \nu \neq p \\ 0 & \text{in the rest} \end{cases}$$

With the currents and the propagators we can calculate the amplitude which is given by the following relation

$$\mathcal{A} = \frac{6\kappa_0^2}{\kappa_\perp^2} T_p^2 V_{p+1}. \quad (138)$$

If we compare the amplitude obtained from string calculations (117) with the amplitude computed from field theory (138) we obtain

$$V_{p+1} \frac{24\pi}{2^{10}} (4\pi^2 \alpha')^{11-p} G_{25-p}(Y) = \frac{6\kappa_0^2}{\kappa_\perp^2} T_p^2 V_{p+1}. \quad (139)$$

The value of the Green's function that enter the l.h.s. of the relation (139) is found in r.h.s. in the momentum space

$$G_{25-p}(Y) = \frac{1}{\kappa_{\perp}^2}. \quad (140)$$

The rest of the terms in (139) give us an equation from which we can determine the tension of the brane

$$\frac{\pi}{2^8 \kappa_0^2} (4\pi^2 \alpha')^{11-p} = T_p^2. \quad (141)$$

We note that there exist a relation between the tension of different branes given by the following relation

$$\left(\frac{T_{p+1}}{T_p} \right)^2 = 4\pi^2 \alpha'. \quad (142)$$

4.3 Exercises

Exercise 1

Compute the integral over p 's in (106).

Exercise 2

Compute the trace in (109).

Exercise 3

Construct the corresponding trace for fermionic oscillators and compute it.

Exercise 4

Show the r.h.s. of (124).

Exercise 5

Find the action (132).

5 Boundary state description of bosonic Dp -branes

The tree level diagram in closed string theory describes the following phenomenon: a closed string is generated from the vacuum, it propagates a certain interval of time and then it is annihilated again in the vacuum. One can sandwich this diagram between two states which will be inserted in the position of the ending circles of the cylinder, i. e. on the boundary of the world-sheet. Such of states that describe the creation and annihilation of the closed strings are called **boundary states**. In the previous paragraph we encountered cylinder

diagrams which described the interaction between two Dp -branes at tree level. It is then natural to ask if there is any boundary state that could be interpreted as a Dp -brane? The answer is yes. Such of boundary state represents a microscopic description of the brane in terms of closed string modes [15, 16].

We recall that the open string boundary conditions that define a Dp -brane are given by the following relations

$$\begin{aligned}\partial_\sigma X^a|_{\sigma=0} &= 0, \quad a = 0, 1, \dots, p \\ X^i|_{\sigma=0} &= y^i, \quad i = p+1, \dots, 25\end{aligned}\tag{143}$$

To pass to the closed string boundary condition, one has to interpret the cylinder as tree-level diagram in closed string sector like in the previous section. The relations (143) take the following form

$$\begin{aligned}\partial_\tau X^a|_{\tau=0} &= 0, \quad a = 0, 1, \dots, p \\ X^i|_{\tau=0} &= y^i, \quad i = p+1, \dots, 25.\end{aligned}\tag{144}$$

If we want to interpret the Dp -branes as boundary states, then we must implement the boundary conditions (144) in the Fock space of perturbative closed string. This is done by interpreting the string coordinates as operators

$$\begin{aligned}\partial_\tau X^a|_{\tau=0}|B\rangle &= 0, \quad a = 0, 1, \dots, p \\ (X^i|_{\tau=0} - y^i)|B\rangle &= 0, \quad i = p+1, \dots, 25.\end{aligned}\tag{145}$$

The equation (145) define the boundary state $|B\rangle$. To find its solution we expand the string operators in terms of oscillation modes using the solution of the equations of motion given in Section 2

$$X^\mu(\tau, \sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left[\alpha_n^\mu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \right],\tag{146}$$

which act on the closed string vacuum

$$|0\rangle = |0\rangle_\alpha |0\rangle_{\tilde{\alpha}} |p\rangle.\tag{147}$$

The equations (145) take the following form

$$\begin{aligned}(\alpha_n^a + \tilde{\alpha}_{-n}^a)|B\rangle &= 0 \\ (\alpha_n^i - \tilde{\alpha}_{-n}^i)|B\rangle &= 0 \\ \hat{p}^a|B\rangle &= 0 \\ (\hat{x}^i - y^i)|B\rangle &= 0.\end{aligned}\tag{148}$$

It is worthwhile to note that (148) are not the only conditions that should be imposed on the Hilbert space. Actually, we have to produce *physical* boundary states, and therefore the negative norm state should be excluded from the solutions of (148). This can be achieved by taking into account the BRST invariance of the theory which is encoded in the right and left-moving BRST operators Q and \tilde{Q} , respectively. The BRST invariant state must satisfy

$$(Q + \tilde{Q})|B\rangle = 0 \quad (149)$$

which allows us to factorize the boundary state in

$$|B\rangle = |B_X\rangle |B_{gh}\rangle. \quad (150)$$

The ghost contribution to the boundary states is defined in terms of the modes of ghost and antighost fields by the following equations

$$\begin{aligned} (c_n + \tilde{c}_{-n})|B_{gh}\rangle &= 0 \\ (b_n - \tilde{b}_{-n})|B_{gh}\rangle &= 0. \end{aligned} \quad (151)$$

We are not going to enter into the details of the BRST quantization and we refer the reader to [15].

A simpler system

In order to find the solutions to (148) we take a look at a simpler model, a single oscillator and its copy with the following boundary conditions

$$\begin{aligned} (a \pm \tilde{a}^\dagger)|b\rangle &= 0 \\ (a^\dagger \pm a)|b\rangle &= 0. \end{aligned} \quad (152)$$

We know that the coherent states of the single oscillator satisfy the following relations

$$\begin{aligned} a|\alpha\rangle &= \alpha|\alpha\rangle \\ |\alpha\rangle &= e^{\alpha a^\dagger}|0\rangle. \end{aligned} \quad (153)$$

The first thing we can try is to replace the phase α by an operator that depends on the operator \tilde{a}^\dagger . The simplest phase is this operator itself multiplied by an unknown phase number, and thus the boundary state $|b\rangle$ can be written as

$$|b\rangle = e^{f a^\dagger \tilde{a}^\dagger}|0\rangle. \quad (154)$$

The phase f can be determined by plugging (154) into (153) from which we get $f = \pm 1$.

It is now straightforward to compute the solution to (148). To this end we recall that the creation operators of string modes are given by

$$a_n^{\mu\dagger} = \sqrt{n} \alpha_{-n}^\mu, \quad n > 0 \quad (155)$$

and that the string is actually a collection of oscillators. The last two equations in (148) will just localize the state in the transverse space and thus will give a delta-function factor. The boundary states has the following general form

$$|B_X\rangle = N_p \delta^{25-p}(\hat{x}^i - y^i) \left(\prod_{n=1}^{\infty} e^{-\frac{1}{n} \alpha_{-n} \cdot S \cdot \tilde{\alpha}_{-n}} \right) |0\rangle_\alpha |0\rangle_{\tilde{\alpha}} |p=0\rangle, \quad (156)$$

where N_p is a normalization constant that should be determined. S has the following form

$$S = (\eta^{ab}, -\delta^{ij}). \quad (157)$$

For completeness we write down the ghost contribution ([15])

$$|B_{gh}\rangle = \exp \left[\sum_{n=1}^{\infty} (c_{-n} \tilde{b}_{-n} - b_{-n} \tilde{c}_{-n}) \right] \left(\frac{c_0 + \tilde{c}_0}{2} \right) |q=1\rangle |\tilde{q}=1\rangle \quad (158)$$

where the ghost ground state is defined by the following equations

$$\begin{aligned} c_n |q=1\rangle &= 0, & n \geq 1 \\ b_n |q=1\rangle &= 0, & m \geq 0. \end{aligned} \quad (159)$$

In any physical gauge we do not have to worry about the unphysical degrees of freedom of string theory. An example of such of gauge is the light-cone gauge where one deals only with the physical degrees of freedom of strings at the cost of loosing the Lorentz invariance. In the case of eq. (156) the light-cone gauge implies summation over the 24 transverse directions on which the metric is Euclidean.

Computation of N_p

In order to have a complete knowledge of the Dp -brane state we have to compute the normalization constant N_p . This can be done by comparing the interaction amplitude computed in the closed string channel with the result obtained from the open string channel. In the previous section we obtained the following result in the open string channel

$$\mathcal{A}_{open} = V_{p+1} (8\pi^2 \alpha')^{-\frac{p+1}{2}} \int_0^\infty \frac{d\tau}{\tau} \tau^{\frac{-p+1}{2}} e^{-\frac{Y^2 \tau}{2\pi\alpha'}} [f_1(e^{-\pi\tau})]^{-24}. \quad (160)$$

In the closed string channel we are at one-loop level. When computing the amplitude we have to care only about the propagator of closed strings between two boundary states

$$\mathcal{A}_{closed} = -\frac{1}{2}\text{Tr} \log Z_0 = -\frac{1}{2}\text{Tr} \log D = \langle B_X | D | B_X \rangle. \quad (161)$$

The closed string propagator can be written in complex coordinate on the Euclidean world-sheet as [1]

$$D = \frac{\alpha'}{4\pi} \int_{|z| \leq 1} \frac{d^2 z}{|z|^2} z^{L_0-1} \bar{z}^{\tilde{L}_0-1}. \quad (162)$$

where L_0 and \tilde{L}_0 are the zero mode Virasoro operators for right/left modes of closed string given by

$$\begin{aligned} L_0 &= \frac{\alpha'}{4} \hat{p}^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \\ \tilde{L}_0 &= \frac{\alpha'}{4} \hat{p}^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n. \end{aligned} \quad (163)$$

The propagator (162) has the property that it propagates states that satisfy the mass-shell conditions $(L_0 - 1)|\phi\rangle = (\tilde{L}_0 - 1)|\phi\rangle = 0$.

The amplitude (160) factorizes in a trace over zero modes and a trace over oscillators. Similar computations were performed in the previous section and we have found the following values of the two factors

$$\begin{aligned} A_0 &= V_{p+1} (2\pi^2 \alpha' t)^{\frac{25-p}{2}} e^{\frac{-Y^2}{2\pi\alpha' t}} \\ A_1 &= \prod_{n=1}^{\infty} \left(\frac{1}{1 - |z|^{2n}} \right)^{24}, \end{aligned} \quad (164)$$

where the contribution of the ghosts has already been taken into account. Here, the following notations have been used

$$|z| = e^{-\pi t}, \quad dz d\bar{z} = -\pi e^{-2\pi t} dt d\sigma \quad (165)$$

Now we put everything together and obtain the interaction amplitude in the closed string channel

$$\mathcal{A}_{closed} = N_p^2 V_{p+1} \frac{\alpha' \pi}{2} (2\pi\alpha')^{-\frac{25-p}{2}} \int_0^\infty \frac{d\tau}{\tau} \tau^{12-\frac{p+1}{2}} e^{-\frac{Y^2 \tau}{2\pi\alpha'}} \left[f_1 \left(e^{-\frac{\pi}{\tau}} \right) \right]^{-24}. \quad (166)$$

This result should be compared with (160). To this end we perform a modular transformation $t \rightarrow 1/\tau$

$$\mathcal{A}_{open} = V_{p+1} (8\pi^2 \alpha')^{-\frac{p+1}{2}} \int_0^\infty \frac{d\tau}{\tau} \tau^{12-\frac{p+1}{2}} e^{-\frac{Y^2 \tau}{2\pi\alpha'}} \left[f_1 \left(e^{-\frac{\pi}{\tau}} \right) \right]^{-24}. \quad (167)$$

Finally, by comparing (160) with (167) we obtain the following value of the normalization constant

$$N_p = \frac{T_P}{2}, \quad (168)$$

where T_p is the brane tension obtained in the previous section.

For further details concerning the boundary state approach to the Dp -brane we refer to the pedagogical lecture notes [15, 16, 17]. In the original papers [18, 19] the relation between the normalization of the boundary states and the tension of D -branes was established while in [20] the D -brane states were constructed in the RNS formalism. This approach gives a good control on the D -branes in the limit where it applies. It is useful for microscopic descriptions of branes as is the case of an alternative formulation of D -branes at finite temperature in the framework of thermo field theory proposed in [21, 22, 23].

5.1 Exercises

Exercise 1

Construct the propagator (162). Argue its form.

Exercise 2

Using (163) in (162) obtain (164).

6 Dp -branes in Type II theories

In this section we are going to review basic topics on supersymmetric D -branes. Some of these ideas will be used in the following lectures at this school. However, due to the lack of time and space and because of the complexity of the topics, we are going to be rather quick. We refer the interested students to the very good reviews [4, 2, 15, 16, 24].

6.1 Closed RNS Superstring

The bosonic string theory presented above suffers from some serious drawbacks as the presence of tachyons in the perturbative spectrum and the absence of fermions. One way to introduce the fermions in string theory is through *supersymmetry* which is a symmetry of the original classical theory that transforms the bosons into fermions and vice-versa. From a pragmatic point of view, it is known from field theory that supersymmetry can cure the divergencies. The supersymmetry can be constructed either by supersymmetrizing the world-

sheet fields (Ramond-Neveu-Schwarz) or by constructing the target-space action (Green-Schwarz). The two constructions are equivalent in the light-cone gauge. The equivalence is based on the modular symmetry and is implemented by the Gliozzi-Scherk-Olive projection which projects out of spectrum half of the states of RNS string. What is left is a theory with space-time supersymmetry equivalent to GS string. Through the GSO projection the tachyon is killed so that the vacua of superstrings are stable [1, 2].

Classical closed superstring

The action of the superstring in the RNS formulation is

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma (\partial_{\alpha} X^{\mu} \partial^{\alpha} X_{\mu} - i \bar{\psi}^{\mu} \rho^{\alpha} \partial_{\alpha} \psi_{\mu}), \quad (169)$$

where to each bosonic field $X^{\mu}(\sigma)$ it is associated a two-dimensional Majorana fermionic field $\psi_A^{\mu}(\sigma)$ with $A = 1, 2$. The Dirac matrices in two dimensions can be chosen imaginary [1] and by definition they should satisfy the following algebra

$$\{\rho^{\alpha}, \rho^{\beta}\} = -2\eta^{\alpha\beta}. \quad (170)$$

The action (169) has the Poincaré symmetry in target-space, super-Weyl invariance in two-dimensions and supersymmetry. The supersymmetry transformations mix the bosonic and fermionic variables and are given by the following relations

$$\begin{aligned} \delta X^{\mu} &= \bar{\epsilon} \psi^{\mu} \\ \delta \psi^{\mu} &= -i \rho^{\alpha} \partial_{\alpha} X^{\mu} \epsilon, \end{aligned} \quad (171)$$

where ϵ is a infinitesimal constant Majorana spinor in two-dimensions that parametrizes the supersymmetry transformations. The following supercurrent corresponds to the supersymmetry

$$J_{\alpha} = \frac{1}{2} \rho^{\beta} \rho_{\alpha} \psi^{\mu} \partial_{\beta} X_{\mu}. \quad (172)$$

As in the pure bosonic case, the system is subject to constraints. The constraints are the equations of motion for the two-dimensional supergravity fields (graviton and gravitino) if a general world-sheet metric is considered. However, in the superconformal gauge in which the action (169) is written, they should be imposed by hand and they have the following form

$$T_{\alpha\beta} = \partial_{\alpha} X^{\mu} \partial_{\beta} X_{\mu} + \frac{i}{2} \bar{\psi}^{\mu} \rho_{(\alpha} \partial_{\beta)} \psi_{\mu} - \frac{1}{2} \eta_{\alpha\beta} (\partial_{\gamma} X^{\mu} \partial^{\gamma} X_{\mu} + \frac{i}{2} \bar{\psi}^{\mu} \rho^{\alpha} \partial_{\alpha} \psi_{\mu}) = 0 \quad (173)$$

$$J_{\alpha} = 0. \quad (174)$$

Also, the super-Weyl transformations imply that

$$T_{\alpha}^{\alpha} = 0 \quad (175)$$

$$\rho^{\alpha} J_{\alpha} = 0. \quad (176)$$

It is easy to show that the equations of motion from (169) are the two-dimensional wave equation and the Dirac equation, respectively

$$\partial^{\alpha} \partial_{\alpha} X^{\mu} = 0 \quad (177)$$

$$\rho^{\alpha} \partial_{\alpha} \psi^{\mu} = 0. \quad (178)$$

The topology of the string determines the boundary conditions that should be imposed on the bosonic coordinates. For fermionic coordinates, the boundary conditions are determined from both topology of world-sheet and the supersymmetry. For bosonic coordinates, the boundary conditions are the same as in eq.(3) for closed string and eq.(5) for open string, respectively. For fermions, there are two boundary conditions that can be imposed either after the fermions performs a complete period on the closed string or at the two ends of the string for open strings. These boundary conditions simply state that the spin of the fermion can flip. Since we focus on the closed strings, the boundary conditions are given by the following relations

$$\psi^{\mu}(\tau, \sigma + \pi) = +\psi^{\mu}(\tau, \sigma) \quad R \text{ b.c.} \quad (179)$$

$$\psi^{\mu}(\tau, \sigma + \pi) = -\psi^{\mu}(\tau, \sigma) \quad NS \text{ b.c.} \quad (180)$$

One can use the powerful techniques of complex analysis and of conformal field theories in two dimensions to study the superstring if we map the cylinder into the complex plane \mathcal{C}^*

$$z = e^{2(\tau - i\sigma)} = e^w, \quad (181)$$

from which we can see that the boundary conditions can be written in the following form

$$\psi^{\mu}(e^{2\pi i} z) = -\psi^{\mu}(z) \quad R \text{ b.c.} \quad (182)$$

$$\psi^{\mu}(e^{2\pi i} z) = +\psi^{\mu}(z) \quad NS \text{ b.c.} \quad (183)$$

To prove the relations above, one uses the periodicity on the cylinder of the holomorphic (and antiholomorphic) fermions.

Massless spectrum of the closed superstring

In order to identify the excitations of the massless fields, one has to quantize the superstring. One can apply exactly the same methods used for the bosonic string [1, 2]. We are going to use the canonical quantization for pedagogical reasons.

The bosonic coordinates $X^\mu(\sigma)$ were quantized in Section 2. To quantize the fermionic coordinates we note that the left- and right-moving modes on the closed string are independent. Consequently, in the complex plane, the solutions to the Dirac equation decompose into holomorphic and anti-holomorphic parts. The Fourier decomposition of the holomorphic part is given by the following relations

$$\psi^\mu(z) = \sum_{n \in \mathbb{Z}} d_n^\mu z^{-n-1/2}, \quad (R) \quad (184)$$

$$\psi^\mu(z) = \sum_{r \in Z'} b_r^\mu z^{-r-1/2}, \quad (NS) \quad (185)$$

where $Z' = Z + 1/2$. Similar expressions hold for anti-holomorphic part. The holomorphic and anti-holomorphic sectors describe the left- and right-moving modes, respectively. The coefficients of the expansion in (185) are interpreted as operators acting on the Fock space which is given by the tensor product of left- and right modes

$$|\text{left} \rangle \otimes |\text{right} \rangle. \quad (186)$$

The interpretation of coefficients in terms of creation and annihilation operators is given by their algebra which can be obtained from the postulated anti-commutator relations among fermionic fields [1, 2]

$$\{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu} \delta_{m+n,0}, \quad (R) \quad (187)$$

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu} \delta_{r+s,0}, \quad (NS) \quad (188)$$

and similarly for right-moving modes. From the relations above we see that one can have $\delta_{m+n,0} = 1$ in the Ramond sector. Thus, the states $d_0^\mu |0\rangle$ are in the massless representation of the Clifford algebra

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}. \quad (189)$$

This means that *the ground state in the R-sector is a spinor*.

The super-Virasoro of the theory is lost through quantization but one can show that the anomaly that appears can be cancelled if the dimension of the space-time is $D = 10$. In ten-dimensions, the background spinor from the R-sector can have both \pm chiralities and it is a Majorana-Weyl spinor which we denote by $|A^+\rangle$ and $|A^-\rangle$, respectively, where A are spinor indices of $Spin(8)$ transversal group.

The above analysis can be repeated verbatim for the right-moving sector. Due to the tensor product structure of the Fock space (186) we can have background spinors of different chiralities in the two sectors. Therefore, we can classify the closed superstrings in

Theory	Ground state	
Type IIA	$ A^+\rangle \otimes \bar{A}^-\rangle$	(190)
Type IIB	$ A^+\rangle \otimes \bar{A}^+\rangle$	

The Type IIA is not chiral, i.e. the vacua in the two sectors have opposite chirality while the Type IIB theory is chiral since the two vacua has the same chirality. (The sign between the two spinors is relative.)

We can quantize the system along the same line as in the bosonic case, but we are not going to present the details here. They can be found in the textbooks [1, 2]. The mass operators in the holomorphic sector, necessary to classify the spectrum of the superstring, are given by the following relations

$$\frac{1}{4}M^2 = \sum_{n>0} \alpha_{-n}^i \alpha_n^i + \sum_{r>0} r b_{-r}^i b_r^i - \frac{1}{2}, \quad (R) \quad (191)$$

$$\frac{1}{4}M^2 = \sum_{n>0} \alpha_{-n}^i \alpha_n^i + \sum_{n>0} n d_{-n}^i d_n^i, \quad (NS) \quad (192)$$

where $\alpha' = 1$ and the indice $i = 1, 2, \dots, 8$ labels the transverse directions (light-cone gauge.) The one-half factor from the NS -sector comes from the normal ordering of the super-Virasoro operator. In the R -sector its value is zero.

The states are constructed by acting with the creation operators on the vacuum. We are interested in space-time supersymmetric states which is the observed supersymmetry, rather than in the world-sheet supersymmetric states. To obtain the target-space spectrum, we have to perform the GSO projection onto the world-sheet vector space. This projection is represented by the GSO operator $(-)^F$ under which the bosonic fields X^μ are even and the fermionic ones ψ^μ are odd

$$[(-)^F, X^\mu] = 0, \quad \{(-)^F, \psi^\mu\} = 0. \quad (193)$$

These properties determine the operator F up to a sign which is fixed by asking that in the open superstring spectrum the photon be invariant under the GSO projection.

The GSO operator is represented by

$$\Gamma = \Gamma^0 \dots \Gamma^8 (-)^{\sum_{n>0} d_{-n}^i d_n^i}, \quad (R) \quad (194)$$

$$G = (-)^{F+1} = (-)^{\sum_{r>0} b_{-r}^i b_r^i + 1}, \quad (NS) \quad (195)$$

in the two sectors of holomorphic spectrum. Here, Γ^i are the Dirac matrices in $D = 10$ dimensions. With these conventions, the states that are odd under the action of the GSO operators are projected out.

In the light cone gauge the massless spectrum of the closed superstrings is given by the product between the left- and right-moving states classified according to the three representations of the $SO(8)$ group (the little group of $SO(1,9)$), namely $\mathbf{8}_V$, $\mathbf{8}_+$, $\mathbf{8}_-$ as follows

$$(\mathbf{8}_V \oplus \mathbf{8}_+)_L \otimes (\mathbf{8}_V \oplus \mathbf{8}_\pm)_R, \quad (196)$$

where the subscripts **l** and **r** stand for left- and right-moving modes, respectively.

Type IIA massless spectrum

The non-chiral closed superstring has the following spectrum

$$\begin{aligned} (\mathbf{8}_\mathbf{v} \oplus \mathbf{8}_+)_{\mathbf{l}} \otimes (\mathbf{8}_\mathbf{v} \oplus \mathbf{8}_-)_{\mathbf{r}} = & (\mathbf{1} + \mathbf{28} + \mathbf{35}_\mathbf{v})_{NS-NS} \oplus (\mathbf{8}_\mathbf{v} + \mathbf{56}_\mathbf{v})_{R-R} \\ & \oplus (\mathbf{8}_+ + \mathbf{56}_-)_{NS-R} \oplus (\mathbf{8}_- + \mathbf{56}_+)_{R-NS}, \end{aligned} \quad (197)$$

The states from $NS - NS$ and $R - R$ sectors are bosonic since they are either a product of two bosonic states, or a product of two spinor states while the states from the $NS - R$ and $R - NS$ sectors are fermionic being products of a bosonic and a fermionic state. The number of bosonic and fermionic degrees of freedom match and this is the first indication of the existence of supersymmetry [1]. The numbers in the brackets indicate the irreducible representations of the $SO(8)$ group. According to it, the bosonic states can be identified with the excitations of the dilaton, Kalb-Ramond field and gravitational potential $\phi, B_{\mu\nu}, g_{\mu\nu}$ in the $NS - NS$ sector and with the excitations of an one-form and a three-form fields A_μ and $A_{\mu\nu\rho}$, respectively, in the $R - R$ sectors.

Type IIB massless spectrum

The chiral closed superstring has the following spectrum

$$\begin{aligned} (\mathbf{8}_\mathbf{v} \oplus \mathbf{8}_+)_{\mathbf{l}} \otimes (\mathbf{8}_\mathbf{v} \oplus \mathbf{8}_-)_{\mathbf{r}} = & (\mathbf{1} + \mathbf{28} + \mathbf{35}_\mathbf{v})_{NS-NS} \oplus (\mathbf{1} + \mathbf{28} + \mathbf{35}_+)_{R-R} \\ & \oplus (\mathbf{8}_- + \mathbf{56}_+)_{NS-R} \oplus (\mathbf{8}_- + \mathbf{56}_+)_{R-NS}, \end{aligned} \quad (198)$$

The bosonic fields that correspond to the massless irreducible representations in the Type IIB theory are the dilaton, Kalb-Ramond field and gravitational potential $\phi, B_{\mu\nu}, g_{\mu\nu}$ in the $NS - NS$ sector and a scalar field, a two-form field and a self-dual four-form field $\chi, A_{\mu\nu}$ and $A_{\mu\nu\rho\sigma}^+$ in the $R - R$ sector. The self-duality of the four form field means that the field strength is equal to its Hodge dual.

As we can see from (197) and (198) above, the graviton appears in the two theories. Also, by GSO-projection the tachyon has been removed since it is an odd state under the action of $(-)^F$ operator. The theories are free of anomalies in $D = 10$, free of tachyons and contain fermions. Beside the fundamental interactions, some other interactions mediated by p -form fields are predicted. In four dimensional space-time such of interactions cannot be written since there is not enough room to accomodate the higher rank p -forms. Indeed, a three-form in four dimensions has the same number of components as a two-form, and a four-form as a one-form or a vector. Thus, no new gauge potentials can be constructed. The Type II theories have $N = 2$ supersymmetry in $D = 10$, corresponding to the two generators of opposite or equal chirality.

Let us note in the end that there are three more string theories known as Type I, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$. The Type I theory contains open strings. Therefore, it has just $N = 1$ supersymmetry. The heterotic theories are supersymmetric just in one of the sectors (left or right). The dilaton, Kalb-Ramond and gravitational fields are common to all these theories, but the p -forms differ from one theory to another. For more details, we refer the reader to [1, 2, 4].

Spin operators and space-time supercharges

All the nice features of the superstrings come from the new symmetry, the supersymmetry that has been introduced in (171). The supersymmetry is implemented at the quantum level by the supercharge operators.

Let us review how they can be constructed [1, 2]. In the complex world-sheet variables, the superstring theories are described by **superconformal field theories** (SCFT). These field theories are well known. They have nicer properties than field theories in four dimensions since there are an infinite number of symmetries in two-dimensions (Virasoro) that give a good control of the S -matrix. Using the techniques of SCFT, one can show that the spinor ground states in the R -sector can be obtained by acting with some operators called **spin operators** $S_a^\pm(z)$ and $\bar{S}_a^\pm(\bar{z})$ on the vacuum of the NS -sector in both left- and right-moving sectors [2, 4]. The spin operators transform as space-time spinors which justify their names and there are 32 of them. One fundamental object in conformal field theory is the **operator product expansion** (OPE) which encodes all the information about the theory since it is equivalent with the commutation relations. For the S and \bar{S} fields, the basic OPE's are with the fermionic fields

$$\psi^\mu(z)S(w) \sim (z-w)^{-\frac{1}{2}}\Gamma^\mu S(w) \quad (199)$$

$$\bar{\psi}^\mu(\bar{z})\bar{S}(\bar{w}) \sim (\bar{z}-\bar{w})^{-\frac{1}{2}}\Gamma^\mu \bar{S}(\bar{w}), \quad (200)$$

where Γ^μ are the Dirac matrices and \sim means that the irregular terms are discarded. The contour integral of fermion-emission operators are just the supercharges

$$Q = \oint \frac{dz}{z} S(z), \quad \bar{Q} = - \oint \frac{d\bar{z}}{\bar{z}} \bar{S}(\bar{z}). \quad (201)$$

This way of understanding the supercharges will be useful later when we will analyse the supersymmetries preserved by the D -branes.

6.2 Type II supersymmetric D -branes

As in the case of the bosonic string theory, the D -branes in the superstring theory are defined by a mixed Neumann and Dirichlet boundary conditions on the open super-

string world-sheet. The fermionic boundary conditions should be compatible with the supersymmetry (171). If we pass to complex coordinates on the Euclidean complex plane $z = \exp(\tau + i\sigma)$ the whole set of boundary conditions can be written as follows [2, 24]

$$\begin{aligned}\partial X^a &= \bar{\partial} X^a|_{Im\, z=0} \\ \partial X^i &= -\bar{\partial} X^i|_{Im\, z=0}\end{aligned}\tag{202}$$

for bosonic coordinates, and

$$\psi^a = \bar{\psi}^a|_{Im\, z=0}, \quad \psi^i = -\bar{\psi}^i|_{Im\, z=0}, \quad (R)\tag{203}$$

$$\psi^a = -\bar{\psi}^a|_{Im\, z=0}, \quad \psi^i = \bar{\psi}^i|_{Im\, z=0}, \quad (NS)\tag{204}$$

for fermionic coordinates, where $a = 0, 1, \dots, p$ and $i = p + 1, \dots, 9$.

In general, the presence of such of extended objects in superstring theory will break the original symmetries. We saw that in the bosonic theory where the Poincaré symmetry was broken. In the supersymmetric case D -brane breaks the space-time symmetry down to $SO(1, p) \times SO(9 - p)$. We may ask what happens with the supersymmetry?

In order to have some conserved supersymmetry we need supercharges that leave the vacuum invariant, or equivalently, spin fields. Since they satisfy the OPE (200) it is easy to see that at the boundary there will be some relations that should be imposed on the spin fields in order to maintain the compatibility between the boundary conditions (204) and the OPE (200). These relations represent the boundary conditions for spin fields and one can show that they transform S into \bar{S} and vice-versa [2, 24]

$$S = \Pi_{(p)} \bar{S}.\tag{205}$$

One can look for an operator $\Pi_{(p)}$ that is constructed from Dirac matrices since the spin operators transform as spinors in $D = 10$. By introducing (205) into (200) one can show that $\Pi_{(p)}$ should satisfy the following relations [2, 24]

$$[\Pi_{(p)}, \Gamma^a] = 0, \quad \{\Pi_{(p)}, \Gamma^i\} = 0.\tag{206}$$

Such of operator exists, and has the following form

$$\Pi_{(p)} = i^{9-p} \Gamma_{11} \Gamma^{p+1} \Gamma_{11} \Gamma^{p+2} \dots \Gamma_{11} \Gamma^9,\tag{207}$$

where

$$\Gamma_{11} = \Gamma^0 \Gamma^1 \dots \Gamma^9.\tag{208}$$

Since $\Pi_{(p)}$ maps left spin operators to right spin operators, it should flip the chirality for Type IIA spin operators and leave it invariant for Type IIB spin operators. This chirality is flipped for p even, and left unchanged for p odd. Therefore, we have the following

Theory	Dp -branes	
Type IIA	$p = 0, 2, 4, 6, 8$	(209)
Type IIB	$p = -1, 1, 3, 5, 7, 9$	

The $p = -1$ brane makes sense only in the Euclidean space-time where it is interpreted as a soliton. $p = 9$ is a degenerate case in which the string can propagate freely in the bulk of space-time and it is consistent only in Type I theory where some auxiliary construction should be done.

We can see in this way that the Dp -brane breaks the supersymmetry of the background and only half of it is preserved. Some other configurations of BPS-branes can be imagined which break the supersymmetry to $1/2$, $1/4$, ... of the original number of supercharges. There are also non-BPS branes which do not preserve any supersymmetry at all, but discussing these topics is out of the scope of these lectures (see [2, 4, 15, 16, 24].)

6.3 Some properties of the D -branes

The supersymmetric D -branes can be treated in the similar fashion as the bosonic ones studied in the previous sections. Besides their tension, they are characterized by other physical quantities as RR charges and supersymmetry. We are going to review the basic properties of BPS-branes in what follows.

Tension and charge of D -brane

The tension of the brane is computed from the exchange of the massless modes of the closed strings. The difference from the bosonic case lies in the fact that there is a two-form field in the RR sector whose excitations should be taken into account. The details of the computations are given in [2] (see also [4, 24]) and the result is

$$T_p^2 = \frac{\pi}{K^2} (4\pi^2 \alpha')^{3-p}, \quad (210)$$

where K is the Newton's constant in ten dimensions. The tension of the brane equals its RR charge-density e_p . The RR p -form field couples with the brane as the point-like particle couples with the gauge potential in four dimensions. The coupling can be electric-like or magnetic-like, i. e. with the strength-form field or with its dual. For example, the electric-like interaction term is given by the Wess-Zumino action

$$e_{p+1} \int d^{p+1} \xi \hat{A}^{p+1} \quad (211)$$

in the simplest situation when the topology is kept simple. Here, the integral is over the world-volume of the brane and A^{p+1} is the pull-back of the RR field on the world-volume

of the brane. e_p is the *RRcharge* of the D -brane and can be calculated by integrating the dual of the field strength on a sphere in the transverse space around the brane

$$e_{p+1} = \int_{S^{8-p}} *F_{8-p}. \quad (212)$$

The magnetic charge can be computed similarly as

$$g_{7-p} = \int_{S^{p+2}} F_{p+2}. \quad (213)$$

In Eq.(212) and Eq.(213) the rank of the forms have been written explicitly. We recall that in order to perform the integration, the dimension of the manifold on which we integrate and the rank of the integrated form should be equal (see also K. Stelle's lecture notes at this school.) The electric and magnetic charges of the D -branes can be quantized following Dirac's prescription

$$\frac{e_{p+1}g_{7-p}}{4\pi} = \frac{n}{2}, \quad (214)$$

where n is an integer number.

It is important to note that the parallel BPS-branes do not feel any force among them since the contribution of the NSNS and RR sectors is equal and of opposite sign. However, for branes at angles the situation is different and for some values of relative angles and distances tachyons appear in the system [28].

Effective action

The effective action of the BPS D -branes can be calculated by using the same method as in the bosonic case. The difference comes from the RR field which for a hyperplanar static brane has only one component coupling with the brane. However, this action would describe only the bosonic sector of the theory. In order to obtain a supersymmetric low energy action, one has to generalize it to include the space-time supersymmetry. The supersymmetric action was obtained in [37]. There is some subtlety involved in this generalization, given by the fact that the fermionic space-time variables are twice in number than necessary. Consequently, one has to impose another local fermionic space-time symmetry called *k-symmetry*, known from the supersymmetric generalization of particles and strings [1, 2]. This symmetry will ensure the correct space-time degrees of freedom for the fermionic coordinates.

The bosonic part of the effective action contains two terms called Dirac-Born-Infeld action and Wess-Zumino action which have the following forms

$$S_{DBI} = T_p \int d^{p+1}\xi e^{-\hat{\phi}} \left[-\det(\hat{G}_{\alpha\beta} + \hat{B}_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}) \right]^{\frac{1}{2}} \quad (215)$$

$$S_{WZ} = T_p \int d^{p+1}\xi \hat{A} \wedge e^{2\pi\alpha' F} \wedge \left(\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})} \right)^{\frac{1}{2}}. \quad (216)$$

Here, the space-time fields are pulled-back on the world-volume of the D -brane. In the Wess-Zumino action, the exponential should be expanded to saturate the dimension of the integral. This is because the fields are expressed by differential forms. The higher dimensional terms give zero contribution. The Dirac-Born-Infeld term has properties similar to the effective action of the bosonic D -brane. As was discussed in Section 3, it generalizes the geometric action, i. e. that is proportional to the volume of brane trajectory, to a background with non-vanishing fields. The generalization of DBI-action to non-abelian potentials $A^a(\xi)$ is not understood yet.

Let us discuss the Wess-Zumino term. It is interpreted as the term that generalizes the coupling of the brane with the RR $(p+1)$ -form fields and should be calculated by expanding the exponential as discussed above. The objects $\mathcal{A}(\mathcal{N})$ and $\mathcal{A}(\mathcal{T})$ are topological invariants that characterize the tangent bundle over the space-time manifold, decomposed into the normal bundle \mathcal{N} to the brane world-volume and the tangent bundle \mathcal{T} to it. This invariant is called *roof-genus* or *Dirac genus* of the bundle and it is defined in terms of the Chern classes. For a vector bundle E , the definition of the roof-genus is

$$\mathcal{A}(E) = \prod_n \frac{\lambda_n/2}{\sinh(\lambda_n/2)}, \quad (217)$$

where $\lambda_n = c_1(L_n)$. Here, c_1 is the first Chern class of the bundle L_n which is a line bundle. One can expand the roof-genus in terms of Pontryagin classes

$$\mathcal{A}(E) = 1 - \frac{1}{24}p_1(E) + \cdots, \quad (218)$$

where

$$p_n(E) = (-1)^n c_{2n}(E \otimes_R C) \quad (219)$$

is the n -th Potryagin class of the vector bundle E tensored with the complex numbers field C . More intuitively, the roof-genus can also be expressed in terms of the curvature of the 2-form R

$$\mathcal{A}(E) = 1 + \frac{1}{(4\pi)^2} \frac{1}{12} \text{Tr} R^2 + \cdots \quad (220)$$

which defines the Dirac genus as a sum of invariant polynomials in the curvature form [40].

The Wess-Zumino action generalizes the coupling of a point-like charge with a gauge potential to a higher dimensional object coupled to the $(p+1)$ -form. Since the D -brane is an extended object, it may assume various topologies in a given background. Therefor, the coupling terms should be topological invariants to guarantee that the formulation of the theory does not change when going from one topology to another. This is how the topological invariants in the action can be roughly explained. In the case of a point-like particle we cannot see all these complications due to the trivial topology of the particle.

6.4 Exercises

Exercise 1

Prove (183) starting from (180).

Exercise 2

Show that, for the superstring in trivial background, the GSO operator $(-)^F$ projects out the tachyon from the spectrum.

Exercise 3

Argue that the electric and magnetic charges of a D -brane satisfy the quantization condition (214).

7 Discussions

In these notes we have argued that there are extended objects in string theory called D -branes which exhibit, beside a geometric structure, physical properties as tension and, for supersymmetric branes, charges. We have derived the effective action of the bosonic D -branes and we have given a microscopic description of them. Also, we have briefly mentioned some of the properties of the supersymmetric BPS-branes.

The material presented in these lectures is in some sense “classic”. We have not discussed any of the more advanced and new results, part because of the extended background material needed to understand these topics which is unfamiliar to many students and partly because of the lack of time. However, there are some exciting ideas which we are going to review briefly now.

BPS D-brane dualities

We have seen above that the D -branes appear in three of the five string theories. In Type IIA and Type IIB, the dimension of the world-volume of the brane is odd, respectively even. However, by compactifying the world-volume of, say, a Type IIA Dp -brane, on a circle of radius R and taking the limit $R \rightarrow 0$, one obtains a manifold with the dimension p . If this manifold is identified with the world-volume of a $D(p-1)$ -brane we have a map from Type IIA branes to Type IIB branes. This is a basic way to establish relations among string theories by using branes, relations called *dualities*. Actually, there are technical details in constructing the dualities. There are several types of them and in the last years there have been done many works in this field [41].

World-volume action of BPS D-branes

In Section 3 we derived the low energy limit of the action of bosonic D -branes and

in Section 6 we discussed its generalization to supersymmetric branes. As was already mentioned, it is not very clear how to generalize the action to non-abelian gauge potential. This is an important line of research, and works have been done recently (for a review see [43].) The problems are related to ambiguities in the definition of the expansion of the determinants of strength-tensor for non-abelian fields. Also, it is not known how to construct a polynomial and local action for D -branes that host a self-dual form field on the world-volume.

Most of the studies have been devoted to low energy limit of the D -branes. The higher energy form of the action for them is unknown.

Non-commutativity

When the Born-Infeld action is generalized to N parallel Dp -branes, the coordinates on the world-volume of the branes become non-commutative functions. This motivated many works on non-commutativity of both branes and strings [44]. The gauge field on the world-volume is $U(N)$ non-commutative gauge theory and the corresponding low energy action is a non-commutative Born-Infeld action. The implications of non-commutativity of space-time physics are currently under intense investigation.

Non-BPS branes

In the last two years there has been an increasing interest in the D -branes that do not preserve any supersymmetry of the background (see the following pedagogical reviews [35, 33, 34]. In brane-antibrane pair (for more than one pair and non-parallel branes see [36]) there is a tachyonic field that cannot be eliminated through the usual GSO projection. Its effective potential displays a local minimum in which it was conjectured that the system reaches a stable state. The evolution to this state is called decayment. By this process non-BPS branes can be obtained from brane-anti-brane pairs and vice-versa, but the dimension of the branes at the beginning and at the end of the process are different. This is a new type of interaction between branes. Since the tachyons are off-shell states, the best tool to investigate the decayment is string field theory. (For references on tachyon physics in D -brane theory see [45].)

Classification of branes

The study of non-BPS branes led to some unexpected applications of mathematics: the classification of brane charges was shown to be given by the topological K-theory of the fibre bundles [48]. This construction was extended to M-theory in [52, 53]. Also, a tentative to include the massive branes in the K-theory framework was done in [54]. More recently, more general D -brane solutions suggested the derived categories as the most appropriate framework for describing the brane charges and decayment [55]. Treating D -branes within the framework of string field theory suggests more algebraic structure behind brane physics.

We cannot end this section without mentioning two revolutionary results in theoretical

physics introduced by D -branes: the Maldacena's conjecture and the sub-millimeter extra dimensions. The first one represents a first explicit proposal and tool of mapping between field theory and gravity. The second theory proposes a solution to the hierarchy problem based on the idea that the Standard Model is localized on a three brane while the gravity lives in the bulk of a five dimensional space-time (see, for example [57, 58, 60] and [61, 62].)

There are many more things that could be said about D -brane theory and many other interesting topics that have been investigated recently. The physics of D -branes is far from being understood, but it is clear that D -branes have been helping us to reveal some of the structure of the most interesting models of the high energy physics and it is likely that their role will not be less important in future.

References

- [1] M. B. Green, J. Schwarz, E. Witten, *Superstring Theory*, (Cambridge University Press, 1989)
- [2] J. Polchinski, *String Theory*, (Cambridge University Press, 1999)
- [3] N. Berkovits, lectures at this school
- [4] C. V. Johnson, hep-th/0007170.
- [5] E. Kiritsis, hep-th/9709062.
- [6] A. Abouelsaood, C. G. Callan, C. R. Nappi and S. A. Yost, Nucl. Phys. B **280**, 599 (1987).
- [7] J. Polchinski, Phys. Rev. Lett. **75**, 4724 (1995) [hep-th/9510017].
- [8] L. Alvarez-Gaume and M. A. Vazquez-Mozo, hep-th/9212006.
- [9] A. A. Tseytlin, A. A. Tseytlin, J. Math. Phys. **42**, 2854 (2001) [hep-th/0011033].
- [10] C. G. Callan, E. J. Martinec, M. J. Perry and D. Friedan, Nucl. Phys. B **262**, 593 (1985). SI ALTH TOTI! Chiara Nappi, etc.
- [11] C. G. Callan, C. Lovelace, C. R. Nappi and S. A. Yost, Nucl. Phys. B **288**, 525 (1987).
[12]
- [12] C. G. Callan, C. Lovelace, C. R. Nappi and S. A. Yost, Phys. Lett. B **206**, 41 (1988).
- [13] C. G. Callan, C. Lovelace, C. R. Nappi and S. A. Yost, Nucl. Phys. B **308**, 221 (1988).
- [14] L. Alvarez-Gaume, D. Z. Freedman and S. Mukhi, Annals Phys. **134**, 85 (1981).
- [15] P. Di Vecchia and A. Liccardo, hep-th/9912161.

- [16] P. Di Vecchia and A. Liccardo, hep-th/9912275.
- [17] M. R. Gaberdiel, Class. Quant. Grav. **17**, 3483 (2000) [hep-th/0005029].
- [18] M. Frau, I. Pesando, S. Sciuto, A. Lerda and R. Russo, Phys. Lett. B **400**, 52 (1997) [hep-th/9702037].
- [19] P. Di Vecchia, M. Frau, I. Pesando, S. Sciuto, A. Lerda and R. Russo, Nucl. Phys. B **507**, 259 (1997) [hep-th/9707068].
- [20] M. Billo, P. Di Vecchia, M. Frau, I. Pesando, S. Sciuto, A. Lerda and R. Russo, R-R states", Nucl. Phys. B **526**, 199 (1998) [hep-th/9802088]
- [21] I. V. Vancea, Phys. Lett. B **487**, 175 (2000) [hep-th/0006228].
- [22] M. C. Abdalla, A. L. Gadelha and I. V. Vancea, Phys. Lett. A **273**, 235 (2000) [hep-th/0003209].
- [23] M. C. Abdalla, A. L. Gadelha and I. V. Vancea, hep-th/0104068.
- [24] C. P. Bachas, hep-th/9806199.
- [25] J. H. Schwarz, hep-ex/0008017.
- [26] A. Sen, Nucl. Phys. Proc. Suppl. **94**, 35 (2001) [hep-lat/0011073].
- [27] J. H. Schwarz, hep-th/9908144.
- [28] S. P. de Alwis, Phys. Lett. B **461**, 329 (1999) [hep-th/9905080].
- [29] M. M. Sheikh Jabbari, Phys. Lett. B **420**, 279 (1998) [hep-th/9710121].
- [30] H. Arfaei and M. M. Sheikh Jabbari, Nucl. Phys. B **526**, 278 (1998) [hep-th/9709054].
- [31] H. Arfaei and M. M. Sheikh Jabbari, Phys. Lett. B **394**, 288 (1997) [hep-th/9608167].
- [32] P. K. Townsend, hep-th/9901102.
- [33] A. Sen, JHEP **9808**, 010 (1998) [hep-th/9805019].
- [34] A. Sen, hep-th/9904207.
- [35] A. Lerda and R. Russo, Int. J. Mod. Phys. A **15**, 771 (2000) [hep-th/9905006].
- [36] I. V. Vancea, JHEP **0104**, 020 (2001) [hep-th/0011251].
- [37] M. Aganagic, C. Popescu and J. H. Schwarz, Phys. Lett. B **393**, 311 (1997) [hep-th/9610249].
- [38] P. K. Townsend, hep-th/0004039.
- [39] P. K. Townsend, hep-th/9712004.
- [40] H. B. Lawson, Jr. and M.-L. Michelson, *Spin Geometry*, (Princeton University Press, 1989)

- [41] P. C. West, hep-th/9811101.
- [42] M. J. Duff, hep-th/9912164.
- [43] A. A. Tseytlin, hep-th/9908105.
- [44] N. Seiberg and E. Witten, JHEP **9909**, 032 (1999) [hep-th/9908142].
- [45] A. Sen, Class. Quant. Grav. **17**, 1251 (2000).
- [46] A. Sen, *Prepared for 29th International Conference on High-Energy Physics (ICHEP 98), Vancouver, British Columbia, Canada, 23-29 Jul 1998.*
- [47] A. Sen, hep-th/9904207.
- [48] E. Witten, hep-th/0006071.
- [49] E. Witten, Int. J. Mod. Phys. A **16**, 693 (2001) [hep-th/0007175].
- [50] K. Olsen and R. J. Szabo, Adv. Theor. Math. Phys. **3**, 889 (1999) [hep-th/9907140].
- [51] K. Olsen and R. J. Szabo, Nucl. Phys. B **566**, 562 (2000) [hep-th/9904153].
- [52] D. Diaconescu, G. Moore and E. Witten, hep-th/0005091.
- [53] D. Diaconescu, G. Moore and E. Witten, hep-th/0005090.
- [54] I. V. Vanea, hep-th/9905034.
- [55] C. I. Lazaroiu, JHEP **0106**, 064 (2001) [hep-th/0105063].
- [56] M. R. Douglas, hep-th/0011017.
- [57] J. L. Petersen, Int. J. Mod. Phys. A **14**, 3597 (1999) [hep-th/9902131].
- [58] P. Di Vecchia, Fortsch. Phys. **48**, 87 (2000) [hep-th/9903007].
- [59] K. Skenderis, Lect. Notes Phys. **541**, 325 (2000) [hep-th/9901050].
- [60] P. Di Vecchia, hep-th/9908148.
- [61] N. Arkani-Hamed, S. Dimopoulos, N. Kaloper and J. March-Russell, hep-ph/9903239.
- [62] S. M. Carroll, hep-th/0011110.